Strings in Background Fields and Nonassociative Geometry

Felix Rennecke

Munich 2011

Strings in Background Fields and Nonassociative Geometry

Felix Rennecke

Master Thesis
at the Max–Planck–Institut für Physik
within the Elite Master Program
Theoretical and Mathematical Physics
of Ludwig–Maximilians–Universität
München

handed in by
Felix Rennecke
from Hamburg

Munich, upon completion

MAX-PLANCK-GESELLSCHAFT

First supervisor: Priv.–Doz. Dr. Ralph Blumenhagen
Second supervisor: Prof. Dr. Dieter Lüst

Abstract

In this master thesis some aspects of the behavior of bosonic strings in the presence of background fields are studied. The main focus will be the implications on the geometry of the target space manifold and the structure of the associated conformal field theory (CFT). At first we review the consequences of quantum consistency of the theory in a geometric manner, explaining the equations which govern viable string backgrounds. We discuss Wess-Zumino-Witten (WZW) models and the underlying two dimensional CFT. The effect of T-duality transformations on background fields will be explained and the resulting geometries illustrated. Then we review open bosonic string theory in the presence of a constant Kalb-Ramond field and elaborate on the emerging non-commutative geometry on D-branes by discussing the so-called Moyal star-product and CFT correlations functions. The main part of the thesis will be the discussion of closed bosonic strings with a constant H-flux at $\mathcal{O}(H)$ respectively R-flux, which is the background suggested after performing three formal T-dualities. We develop the CFT by determining the algebra of holomorphic currents using conformal perturbation theory and compute tachyon correlation functions. The crucial observation is that whereas the H-flux has a rather weak impact on the theory, the R-flux gives rise to a noncommutative and nonassociative target space geometry. We conclude by identifying the origin of the nonassociativity in the CFT and by constructing an N-ary product for functions on the spacetime associated to the R-flux capturing the structure of the CFT correlation functions.

Contents

Chapter 1

Introduction

The appearance of geometries which are not ”point-like“ is well-known since quantum mechanics. The Heisenberg uncertainty relation $\Delta x\,\Delta p \geq \frac{\hbar}{2}$ introduces cells instead of points on the phase space for which x and p are coordinates and thus makes the conception of a space as consisting of points obsolete in quantum theory. Furthermore, quantum field theory suffers from not being finite in the ultraviolet regime in most instances and it was suggested already very early that a noncommutative structure of the spacetime at small length scales could provide an effective ultraviolet cut-off [1]. Although the different path taken by renormalization proved to be very useful to cope with these difficulties for gauge theories, gravity still eludes this description and may necessitates reconsidering cut-offs due to a noncommutative geometry. From a physical point of view, this motivates concerning with noncommutative geometry [2, 3] in the context of quantum field theory [4].

Studying noncommutative geometry amounts also to study the functions on the corresponding manifold since in certain cases, the topology of the manifold can be recovered completely from the algebra of functions on the space [2].

String theories provide a unified framework to address these questions as it contains quantum gauge theories as well as a consistent quantum gravity. Usually, string theories are described by two-dimensional nonlinear sigma models where a background spacetime geometry is specified whose features are probed by the string moving in it. Gauge theories arise from open string theories – the end-points of open strings are restricted to hypersurfaces in spacetime called D-branes on which the gauge theories are realized – while gravity is due to closed strings. Consistently quantizing string theory imposes constraints on the backgrounds like an extension of Einstein's equations and the requirement for the spacetime to be usually more than four-dimensional. The admissible backgrounds can be described as parallelizable manifolds in Riemann-Cartan geometry [5]. If all these constraints are satisfied, the sigma model describes a quantum conformal field theory (CFT) whose properties are crucial for studying string theory. In particular, it turns out that the string is a very different probe for the spacetime than a point-particle. For instance, the surplus spacetime dimensions are condensed in tiny compact spaces which is known in field theory as Kaluza-Klein reduction. However, in string theory the compactification behaves differently since the string can, as opposed to a point particle, also wind around compact directions. One of the most interesting phenomena which can be traced back to this property is the so-called *T-duality* (reviewed in [6]). It states that a string cannot distinguish certain types of geometries, i.e. although one obtains very different spacetime geometries, the quantum theories remain the same [7].

Very intriguingly, the geometry on D-branes in open string theory was found to be noncommutative in the presence of a constant B-field [8, 9]. This was reformulated as noncommutative gauge theory on D-branes [10] with the usual product for fields replaced by the noncommutative Moyal star-product. Thus the before mentioned interesting geometries are realized in string theory, at least for gauge theories. In this context it is natural to ask about the analogous situation for closed strings, i.e. gravity which will be addressed in this thesis.

Conformal field theories describing the closed string in background fields are given by so-called *Wess-Zumino-Witten* (WZW) models [11]. In the context of these, [12] found the spacetime coordinates to be not only noncommutative but also *nonassociative*. This was simultaneously also confirmed in [13] by taking a different approach and emphasizing the significance of T-dual geometries involved. However, the origin of nonassociativity remained largely unclear.

T-duality provides a fertile tool for studying the properties of string theory. In particular, it showed the significance of D-branes [14] to mention a famous example. It also gives rise to very interesting geometries which can for instance be found by applying T-duality to a flat space with non-vanishing three-form flux $H = dB$ [15, 16, 17]. This in particular reveals the appearance of so-called *non-geometries* as string backgrounds. They are non-geometric in that they cannot be described as usual manifolds. One example is the T-fold [18] which is obtained after performing two T-dualities and is only geometric locally. A third T-duality can only be applied formally and is suspected to yield a background which is not geometric even locally, called *R-flux*. Instead it is supposed to give rise to nonassociative geometry [19], hinting towards a source of the structure found more explicitly in [12, 13].

The purpose of this thesis is to study some of these non-geometric aspects in closed bosonic string theory and in particular to gain some new insights on the poorly understood R-flux as the origin of nonassociative geometry. This is in order to complete the picture we already have for the open string to the bulk, i.e. to the closed string.

The thesis is organized as follows.

- In the second chapter we review the basic notions and conceptions of bosonic strings moving in background fields. First we describe the requirements for a consistent quantum theory introducing the notion of parallelizable manifolds. Then we consider WZW models with a particular focus on the CFT they describe in order to have a tool for dealing with background fields in closed string theory. At last we show how T-duality acts on the background by introducing the Buscher rules and illustrate its effects on a flat three-torus endowed with a constant H-flux.

- The third chapter is devoted to the emergence of noncommutative geometry in open string theory. In particular, we review how to deduce a product on the algebra of functions on D-branes by studying CFT N-point correlation functions and emphasize the properties of such a product.

- In the fourth chapter we would like to generalize this to the closed string in a background given by a constant H-flux and a flat three-torus. This background is admissible, i.e. allows for a CFT description only up to linear order in H. We first describe some classical aspects. Then we develop a CFT for this theory in linear order in the flux, called CFT_H. We will follow the principles we reviewed for WZW models to deduce in particular chiral currents and the energy-momentum tensor. We also determine the current algebra using conformal perturbation theory. We identify the tachyon vertex operator as a physical state in the theory and calculate correlation functions thereof. This is done in the constant H- as well as in the constant R-flux background and it turns out that nonassociative geometry emerges only for the R-flux. The structure of the correlators will allow us to deduce a product on the algebra of functions on the spacetime associated to the R-flux analogously to the open string case and we finally discuss its properties.

The fourth chapter is written on the basis of the recent publication [20].

Chapter 2

Strings in Background Fields

So far there is no background independent formulation of string theory; in the sigma-model description strings can rather be considered as "probing" a given background configuration. The viable backgrounds for the bosonic string are given by the massless states found in the spectrum corresponding to the spacetime metric $G_{\mu\nu}$, the antisymmetric Kalb-Ramond field $B_{\mu\nu}$ and the dilaton ϕ. The massive states would contribute terms incompatible with the desired symmetries of the theory and are therefore not considered.

To approach this problem more systematically we use the symmetries as guiding principle. The starting point for bosonic string theory is a two-dimensional nonlinear sigma model given by the *Polyakov action*

$$\begin{aligned} S_P &= \frac{1}{4\pi\alpha'} \int_\Sigma G_{\mu\nu}(X)\, dX^\mu \wedge \star\, dX^\nu \\ &= \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \,\sqrt{h} h^{\alpha\beta}\, G_{\mu\nu}(X)\, \partial_\alpha X^\mu\, \partial_\beta X^\nu\,, \end{aligned} \tag{2.0.1}$$

where Σ is a two-dimensional manifold called *worldsheet* which is un-/bounded depending on whether we consider open or closed strings respectively. X denotes an embedding $X : \Sigma \to M$ with M the D-dimensional spacetime or target-space manifold and the *bosonic fields* X^μ are given by the pullback of a choice of local coordinates $x^\mu : M \to \mathbb{R}$, $\mu, \nu \in \{0, \ldots, D-1\}$ on M, i.e. $X^\mu = X^* x^\mu : \Sigma \to \mathbb{R}$ are coordinates of the submanifold given by the embedding of the worldsheet into the spacetime. The worldsheet is parametrized by $\sigma^\alpha = \tau, \sigma$, $\alpha, \beta \in \{1, 2\}$ and equipped with an *Euclidean* metric $h_{\alpha\beta}$. The action (2.0.1) admits the following classical symmetries:

- Two-dimensional diffeomorphism invariance
- Two-dimensional Weyl invariance: $h_{\alpha\beta}(\tau,\sigma) \mapsto \exp[2\omega(\tau,\sigma)]\, h_{\alpha\beta}(\tau,\sigma)$

It also admits two-dimensional conformal invariance which is a combination of diffeomorphism- and Weyl invariance[1].

[1]An automorphism f of Σ is a conformal map if the pulled-back metric is conformally equivalent to the original one, i.e. $f^* g = \exp(2\omega)\, g$.

The most general action incorporating all the symmetries mentioned above can be obtained by adding the term

$$S_B = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \, \sqrt{h} \, i\epsilon^{\alpha\beta} \, B_{\mu\nu}(X) \, \partial_\alpha X^\mu \, \partial_\beta X^\nu \tag{2.0.2}$$

to (2.0.1). $B_{\mu\nu}$ is antisymmetric and can be considered as arising from a gauge field on the spacetime, i.e. the integrand of (2.0.2) is given by X^*B with $B \in \Omega^2(M)$. Hence this term reflects a background given by the Kalb-Ramond field. This interpretation reveals another obvious symmetry given by

$$B \to B + d\omega \tag{2.0.3}$$

for $\omega \in \Omega^1(M)$ if $\partial\Sigma = \emptyset$, i.e. for the closed string.

Although customarily added, for the moment we will not comment on the dilaton since the corresponding contribution to the action would break the desired Weyl invariance already classically and can be considered as a higher order correction to be addressed later.

In the following we are interested in string tree-level, i.e. we assume Σ to have genus 0. For the open string this means that Σ has the topology of a disc D^2 and for the closed string it has the topology of a sphere S^2. Thus we are mainly interested in

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \left(h^{\alpha\beta} \, G_{\mu\nu}(X) + i\epsilon^{\alpha\beta} \, B_{\mu\nu}(X) \right) \partial_\alpha X^\mu \, \partial_\beta X^\nu \\ &= \frac{1}{2\pi\alpha'} \int_\Sigma d^2 z \, (G_{\mu\nu}(X) + B_{\mu\nu}(X)) \, \partial X^\mu \, \overline{\partial} X^\nu \, , \end{aligned} \tag{2.0.4}$$

where we have chosen conformal gauge, i.e. $h = \mathrm{diag}(\pm 1, 1)$[2]. We also gave the action rewritten in complex coordinates $z = \exp(\tau - i\sigma)$ with $\partial = 1/(2z)(\partial_\tau + i\partial_\sigma)$ and the respective complex conjugates[3]. This conformally maps the disc to the lower half plane $\mathbb{H}^-$ and the sphere to the compactified complex plane $\mathbb{C} \cup \{\infty\}$. In particular the measure is given by $d^2 z = dz d\overline{z} = 2z\overline{z} \, d\tau d\sigma$.

However, quantum consistency forces us to put some constraints on the choice of background fields which will be discussed in the next section. Then we review a model describing a large class of admissible backgrounds and illustrate the effects of so-called T-Duality on the background fields.

2.1 Background consistency equations

It is a well-accepted maxim that the symmetries apparent in a classical theory should pass on to the quantized theory; when some of them get lost we have to find a way to restore them. It turns out that by quantizing the bosonic string via the path integral,

[2] Depending on whether an Euclidean or Lorentzian signature is suitable.

[3] Note that in $(z, \overline{z})$-coordinates time ordering becomes radial ordering.

manifest Weyl invariance disappears (cf. [21]) which can be characterized as follows. Suppose we perform an infinitesimal Weyl rescaling $\delta h_{\alpha\beta} = \epsilon h_{\alpha\beta}$ on the metric. Varying the action and using the definition of the energy-momentum tensor T gives

$$\delta S = \int_\Sigma d^2\sigma \, \frac{\partial S}{\partial h_{\alpha\beta}} \delta h_{\alpha\beta} \sim \int_\Sigma d^2\sigma \, \sqrt{h} \epsilon \, T^\alpha{}_\alpha \,, \tag{2.1.1}$$

i.e. Weyl invariance of the action requires the trace of T to vanish;

$$\mathrm{Tr}\,(T) = 0 \,. \tag{2.1.2}$$

Unfortunately, it turns out that in quantum theory we obtain

$$\langle \mathrm{Tr}\,(T) \rangle \neq 0 \tag{2.1.3}$$

and thus the Weyl invariance is broken. In other words, the conformal invariance is lost and thus, by renormalization, quantization introduces a scale dependence to the couplings – here G and B – described by the corresponding β-function(al)s. We will elaborate on the specific form of the right-hand-side of (2.1.3) respectively the β-functionals in the following. Requiring the the right-hand-side to vanish then gives the equations for consistent string backgrounds.

2.1.1 Strings on parallelizable manifolds

To calculate the β-functionals encoding the Weyl anomaly it is convenient to use a background field expansion [22]. The basic idea is to expand all the fields in Riemann normal coordinates; then all the terms in this expansion will be manifestly covariant and computations simplify significantly as geodesics appear as straight lines and thus the Levi-Civita connection is just given by the usual derivative. For the expanded action a perturbative expansion in α' can be performed to obtain the renormalized couplings – the metric and Kalb-Ramond field. Computing the renormalized energy-momentum tensor then reveals the Weyl anomaly. For the details of this calculation one may consult [5, 23].

In [5] general nonlinear sigma models of the form (2.0.4) with M given by a group manifold were studied. They describe the renormalization group evolution in a very nice geometric setting on which we want to elaborate in the following. What we denoted B was termed a *torsion potential* for the following reason. Computing the equations of motion for the bosonic fields X of (2.0.4)

$$\begin{aligned} 0 &= \overline{\nabla}_\alpha \, \partial^\alpha X^\mu \\ &= \left(\delta^\mu_\nu \, \partial_\alpha + \Gamma^\mu{}_{\rho\nu} \, \partial_\alpha X^\rho - \frac{i}{2} \, G^{\mu\sigma} \, H_{\sigma\nu\rho} \, \epsilon_{\alpha\beta} \, \partial^\beta X^\rho \right) \partial^\alpha X^\nu \\ &= \left(\nabla_\alpha - \frac{i}{2} \, G^{\mu\sigma} \, H_{\sigma\nu\rho} \, \epsilon_{\alpha\beta} \, \partial^\beta X^\rho \right) \partial^\alpha X^\nu \,, \end{aligned} \tag{2.1.4}$$

where $\Gamma^{\mu}{}_{\nu\rho}$ are the Christoffel symbols for the Levi-Civita connection on the target space M and

$$H = dB \ \ , \text{i.e.} \ \ H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} \, . \tag{2.1.5}$$

For M a group manifold $H_{\mu\nu\rho}$ are proportional to the structure constants of $Lie(M)$ and here they appear as *torsion* on the target space. Indeed, what was suggested in (2.1.4) as a connection on the worldsheet can be seen as the pullback of a metric-compatible connection on the target space M considered as a *Riemann-Cartan* manifold. These are Riemannian manifolds endowed with a torsion tensor $T \in \Gamma(M, \Lambda^2 T^*M \otimes TM)$. In our case the torsion tensor happened to be[4] the three-form H and the relevant geometric notions are as follows.

- A metric-compatible connection on (M, G, H) can be defined by

$$\begin{aligned} \overline{\Gamma}^{\mu}{}_{\nu\sigma} &:= \Gamma^{\mu}{}_{\nu\sigma} - \frac{1}{2} G^{\mu\rho} H_{\rho\nu\sigma} \\ \overline{\nabla}_\mu V_\nu &= \nabla_\mu V_\nu - \frac{1}{2} H_{\nu\mu}{}^{\sigma} V_\sigma \\ \overline{\nabla}_\mu V^\nu &= \nabla_\mu V^\nu + \frac{1}{2} H_{\nu\mu\sigma} V^\sigma \end{aligned} \tag{2.1.6}$$

  for an arbitrary tensor V of rank one.

- Acting on a tensor V of rank one we obtain the algebra of the covariant derivatives

$$\left[\overline{\nabla}_\mu, \overline{\nabla}_\nu\right] V_\sigma = \overline{R}^{\rho}{}_{\sigma\mu\nu} V_\rho + H^{\rho}{}_{\mu\nu} \overline{\nabla}_\rho V_\sigma \tag{2.1.7}$$

  which indeed has the form expected for a general connection.

- The curvature tensor appearing in the above algebra reads

$$\overline{R}_{\mu\nu\sigma\rho} = R_{\mu\nu\sigma\rho} + \frac{1}{2} \nabla_\sigma H_{\mu\nu\rho} - \frac{1}{2} \nabla_\rho H_{\mu\nu\sigma} + \frac{1}{4} H_{\mu\rho\lambda} H^{\lambda}{}_{\nu\sigma} - \frac{1}{4} H_{\mu\sigma\lambda} H^{\lambda}{}_{\nu\rho} \, , \tag{2.1.8}$$

  where R denotes the Riemannian curvature tensor.

- The Ricci tensor obtained by contracting the first and third index of the curvature reads

$$\overline{R}_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} \nabla_\sigma H_{\sigma\mu\nu} - \frac{1}{4} H_{\mu\sigma\rho} H^{\sigma\rho}{}_{\nu} \tag{2.1.9}$$

  upon using the antisymmetry of H.

Coming back to our original goal of studying the β-functionals encoding the Weyl anomaly, Braaten, Curtright and Zachos stated a conjecture in [5] which allows for a more systematic approach to this problem.

[4] Recall that a metric can also be considered as a map $TM \to T^*M$, i.e. raising and lowering an index "dualizes" a tensor in that direction.

Proposition 1. *To all orders in α', the β-functionals of the nonlinear sigma model* (2.0.4) *only depend on the generalized curvature tensors $\overline{R}_{\mu\nu\sigma\rho}$ given in* (2.1.8) *and possibly covariant derivatives thereof.*

This is a generalization of the analogous statement for sigma models (2.0.1) involving the Riemann curvature tensor and has be approved up to second order in α' [5]. Thus upon vanishing of the generalized curvature tensor $\overline{R}$ the Weyl invariance is restored as the β-functionals vanish. To be more precise, at first order in α' the β-functionals for the metric respectively the Kalb-Ramond field are given in terms of the Ricci tensor (2.1.9) as [5]

$$\begin{aligned} \beta^G_{\mu\nu} &= \alpha' \, \overline{R}_{(\mu\nu)} = \alpha' \, R_{\mu\nu} - \frac{\alpha'}{4} \, H_{\mu\sigma\rho} \, H^{\sigma\rho}{}_{\nu} \\ \beta^B_{\mu\nu} &= \alpha' \, \overline{R}_{[\mu\nu]} = \frac{\alpha'}{2} \, \nabla^\sigma H_{\sigma\mu\nu} \end{aligned} \tag{2.1.10}$$

which seems reasonable just by considering the index structure. This observations can be phrased more geometrically.

Definition 1. A n-dimensional manifold M is called *parallelizable* if and only if it admits n global vector fields which are linearly independent everywhere. Equivalently, M is parallelizable if the tangent bundle TM is trivial.

Since a trivial vector bundle admits a flat connection, i.e. one whose associated curvature vanishes, the class of parallelizable manifolds offer consistent string backgrounds[5]. This class consists of group manifolds[6] and S^7, which is the only exception [24] since it is not a Lie group. In summary

Proposition 2. *String backgrounds allowing for a consistent quantum theory in the sense that all classical symmetries carry over during quantization are described by parallelizable manifolds, in particular Lie groups. The equations describing these backgrounds read*

$$\begin{aligned} \alpha' \, R_{\mu\nu} - \frac{\alpha'}{4} \, H_{\mu\sigma\rho} \, H^{\sigma\rho}{}_{\nu} &= 0 \\ \frac{\alpha'}{2} \, \nabla^\sigma H_{\sigma\mu\nu} &= 0 \end{aligned} \tag{2.1.11}$$

up to first order in α'.

2.1.2 The dilaton and the critical dimension

We will now shortly comment on the dilaton, the remaining massless state in the spectrum. The contribution to the action (2.0.4) reads

$$S_D = \frac{1}{4\pi} \int_\Sigma d^2\sigma \, \sqrt{h} \, R^{(2)} \, \phi(X) \,, \tag{2.1.12}$$

[5]Recall that in this picture, the B-field is part of the data of the manifold.

[6]A basis for the associated Lie algebra, i.e. the tangent space at the identity can be made a global basis by moving this local frame to any point of the tangent bundle using the group action. This requires a connected Lie group.

where $R^{(2)}$ denotes the Ricci scalar on the worldsheet. We observe that this term is of higher order of α' compared to (2.0.4) as ϕ has no units. Moreover, it breaks Weyl invariance already classically since the contribution to the energy-momentum-tensor is not traceless. However, due to the α' dependencies the tree-level lack of Weyl invariance can be compensated by a one-loop contribution of the other terms. We will just state the lowest order result for the β-functionals. The complete string path integral also requires the insertion of Faddeev-Popov ghosts by gauge-fixing the worldsheet metric which also have to be considered in the complete calculations. One obtains [25]

$$\begin{aligned} \beta^G_{\mu\nu} &= \alpha' R_{\mu\nu} - \frac{\alpha'}{4} H_{\mu\sigma\rho} H^{\sigma\rho}{}_\nu + 2\alpha' \nabla_\mu \nabla_\nu \phi + \mathcal{O}(\alpha'^2) \\ \beta^B_{\mu\nu} &= \frac{\alpha'}{2} \nabla^\sigma H_{\sigma\mu\nu} + \alpha' \nabla^\sigma \phi \, H_{\sigma\mu\nu} + \mathcal{O}(\alpha'^2) \\ \beta^\phi_{\mu\nu} &= \frac{D-26}{6} - \frac{\alpha'}{2} \Delta\phi + \alpha' \nabla_\sigma \phi \nabla^\sigma \phi - \frac{\alpha'}{24} H_{\mu\nu\sigma} H^{\mu\nu\sigma} + \mathcal{O}(\alpha'^2) \,. \end{aligned} \tag{2.1.13}$$

Hence for a flat, empty space with a constant dilaton we can read-off the critical dimension of the string as $D = 26$. By choosing the background fields appropriately one is also able to obtain consistent strings on arbitrary dimensional manifolds. One may wonder why the critical dimension of the string is not apparent in the analysis of (2.0.4). As the dilaton is associated to the string coupling it actually has to be included in the complete analysis.

2.2 Wess-Zumino-Witten models

The purpose of this section is to introduce a two-dimensional conformal field theory providing a sigma model which describes consistent closed string backgrounds. Thus we assume in the following that the target space is given by a Lie group G. We have already encountered a special case of the model in (2.0.4); one can observe that for a closed string, this action can roughly be rewritten as[7]

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \, G_{\mu\nu}(X) \, \partial_\alpha X^\mu \, \partial^\alpha X^\nu + \frac{i}{4\pi\alpha'} \int_\Sigma X^* B \\ &= \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \, \mathrm{Tr} \left[(\partial_\alpha g)(\partial_\alpha g^{-1}) \right] + \frac{i}{4\pi\alpha'} \int_B \ln(g)^* H \,, \end{aligned} \tag{2.2.1}$$

where $H = dB$ and we have formally rewritten $X = \ln(g)$ for a $g : \Sigma \to G$ and considered the expression as a matrix where the trace is obtained form contraction with G. Moreover, we have introduced a compact three-dimensional manifold B with $\partial B = \Sigma$ which makes sense since $\partial\Sigma = \emptyset$ by assumption and exploited that the pullback commutes with the exterior derivative. In the second term we also extended g to B.

One can imagine that the second term can be given more generally by considering a closed but non-exact three-form as integrand. We will make this more precise in the following.

[7] We choose the Euclidean signature $h_{\alpha\beta} = \delta_{\alpha\beta}$.

2.2.1 The model

We now generalize (2.2.1) according to [11].

Definition 2. Given a compact, bounded three-dimensional manifold B with $\partial B = \Sigma$ and a group manifold G with maps $g : B \to G$, the *Wess-Zumino-Witten* (WZW) model is given by

$$\begin{aligned} S_{\text{WZW}}[g] = &- \frac{k}{2\pi} \int_\Sigma d^2 z \, \text{Tr} \left(\partial \ln g \, \overline{\partial} \ln g \right) \\ &- \frac{ik}{12\pi} \int_B \text{Tr} \left(d \ln g \wedge d \ln g \wedge d \ln g \right) \ , \end{aligned} \tag{2.2.2}$$

where k is some constant. The second, topological term is called *Wess-Zumino*-term. There g is extended to B and the exterior product has to be understood as combination of the exterior product combined with the group structure.

The definitions allows for several remarks.

- To clarify the definitions, suppose t^a , $a \in \{1, \ldots, \dim(\mathfrak{g})\}$ are generators of $Lie(G) = \mathfrak{g}$. Then the trace is normalized such that $\text{Tr}(t^a t^b) = 2\delta^{ab}$ and $[t^a, t^b] = i f_{abc} t^c$ defines the structure constants for the Lie algebra.
- Although the integrand of the topological term looks exact, it is not. The definition of the logarithm is given formally by $d \ln(g) = g^{-1} dg$, hence $d^2 \ln(g) \neq 0$ since $dg \wedge dg$ only vanishes for abelian groups. Therefore the topological term is exact only if G is abelian.
- It can be seen that $d \ln g \wedge d \ln g \wedge d \ln g$ is closed. Thus, by the Poincaré-Lemma, the topological term can be cast into into a form similar to (2.0.4) locally.
- Since the WZW model has a group manifold as target space it describes consistent string backgrounds due to Prop. 2.
- The choice of B such that its boundary gives the worldsheet is of course not unique.

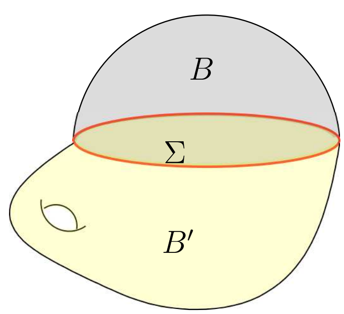

Figure 2.1: The difference between two different choices of manifolds with boundary Σ.

Calculating the difference between a WZ term with B and B' as depicted in Figure 2.1, taking into account the orientation and applying the map g, we obtain a WZ term over a closed compact manifold topologically equivalent to S^3. For a well-defined quantum theory, the difference should not contribute in the path integral and thus has to be a multiple of 2π. By the correct choice of normalization[8] of Tr, these integrals turn out to give $\sim 2\pi k$. Thus, *k has to be an integer*.

- Given a closed $\chi \in \Omega^3(G)$, the WZ term can indeed be considered as $\int_B g^*\chi$.

Let us determine the equations of motion for (2.2.2). It is convenient to use the following statement.

Lemma 1. For $\chi \in \Omega^3(G)$, the variation satisfies

$$\delta \int_B g^*\chi = \int_B \mathcal{L}_{\delta g}\,\chi\,, \tag{2.2.3}$$

where $\mathcal{L}$ denotes the Lie-derivative and δg the vector field tangent to the path of variation at any point.

Proof. By definition of the variation

$$\delta \int_B g^*\chi = \int_B \frac{d}{d\epsilon}(g+\epsilon\delta g)^*\chi\Big|_{\epsilon=0} = \int_B \mathcal{L}_{\delta g}\,\chi\,, \tag{2.2.4}$$

since $g+\epsilon\delta g$ can – by abuse of notation – be considered as the flow of the vector field δg. Then the last step is just the general definition of the Lie-derivative. □

Making use of this lemma, the variation of the WZ term reads

$$\begin{aligned}
\delta_g S^{WZ}_{\mathrm{WZW}} &= -\frac{ik}{12\pi}\int_B \mathcal{L}_{\delta g}\,\mathrm{Tr}\left(d\ln g\wedge d\ln g\wedge d\ln g\right)\\
&= -\frac{ik}{12\pi}\int_B (\iota_{\delta g}\circ d + d\circ\iota_{\delta g})\,\mathrm{Tr}\left(d\ln g\wedge d\ln g\wedge d\ln g\right)\\
&= -\frac{ik}{4\pi}\int_B d\,\mathrm{Tr}\left(\delta_g \ln g\ d\ln g\wedge d\ln g\right)\\
&= -\frac{ik}{4\pi}\int_\Sigma \mathrm{Tr}\left(\delta_g \ln g\ d\ln g\wedge d\ln g\right)\,.
\end{aligned} \tag{2.2.5}$$

In the second step we used the usual identity for $\mathcal{L}$ on forms with ι the interior product, the third exploited that χ is closed and the last step was performed by Stokes theorem. It is remarkable that the variation of the topological term reduces to the worldsheet generally. The variation of the kinetic term is straight-forward and yields

$$\delta_g S^{kin}_{\mathrm{WZW}} = \frac{k}{2\pi}\int_{\partial\Sigma} d^2z\,\mathrm{Tr}\left[\delta_g \ln g\left(\partial\bar\partial\ln g + \bar\partial\partial\ln g\right)\right] \tag{2.2.6}$$

[8] One may consult [26] for details.

upon integration by parts. If we rewrite the variation of the WZ term in local complex coordinates, both results combine to give

$$\delta_g S_{\mathrm{WZW}} = \frac{k}{2\pi}\int_\Sigma d^2z\,\mathrm{Tr}\left(-\delta_g \ln g\;\partial\overline{\partial}\ln g\right) \tag{2.2.7}$$

and the equations of motion therefore read

$$\partial\overline{\partial}\ln g = \partial\left(g^{-1}\,\overline{\partial}g\right) = 0\,. \tag{2.2.8}$$

Now we will devote our attention to the conformal field theory formulated by the WZW model.

2.2.2 Conserved currents

In order to have a genuine CFT the WZW model has to admit two separately conserved holomorphic respectively anti-holomorphic currents. Indeed, following [27], by starting with the usual nonlinear sigma model (2.0.1), the addition of the topological Wess-Zumino term is required in order to obtain these currents. These currents arise as the Noether currents associated to the symmetry of conjugating g with arbitrary group elements. In order to show this we will utilize the following.

Lemma 2. The WZW model (2.2.2) satisfies

$$S_{\mathrm{WZW}}[g_1g_2] = S_{\mathrm{WZW}}[g_1] + S_{\mathrm{WZW}}[g_2] - \frac{ik}{\pi}\int_\Sigma \mathrm{Tr}\left[(g_1^{-1}\,\overline{\partial}g_1)\,(\partial g_2\,g_2^{-1})\right]\,, \tag{2.2.9}$$

called the *Polyakov-Wiegmann* property.

Proof. First we observe that $d\ln(g_1g_2) = g_2^{-1}\,d\ln(g_1)\,g_2 + d\ln(g_2)$. Substituted into the kinetic term yields after some algebraic manipulations

$$\begin{aligned} S^{kin}_{\mathrm{WZW}}[g_1g_2] =& S^{kin}_{\mathrm{WZW}}[g_1] + S^{kin}_{\mathrm{WZW}}[g_2] - \frac{k}{2\pi}\int_\Sigma d^z\,\mathrm{Tr}\left[(g_1^{-1}\,\overline{\partial}g_1)\,(\partial g_2\,g_2^{-1})\right] \\ &- \frac{k}{2\pi}\int_\Sigma d^z\,\mathrm{Tr}\left[(g_1^{-1}\,\partial g_1)\,(\overline{\partial}g_2\,g_2^{-1})\right] \end{aligned} \tag{2.2.10}$$

and for the topological term we obtain

$$\begin{aligned} S^{WZ}_{\mathrm{WZW}}[g_1g_2] =& S^{WZ}_{\mathrm{WZW}}[g_1] + S^{WZ}_{\mathrm{WZW}}[g_2] \\ &- \frac{ik}{4\pi}\int_B \mathrm{Tr}\left[d\ln(g_1)\wedge(d\ln(g_1) + dg_2\,g_2^{-1})\wedge(dg_2\,g_2^{-1})\right] \\ =& S^{WZ}_{\mathrm{WZW}}[g_1] + S^{WZ}_{\mathrm{WZW}}[g_2] + \frac{ik}{4\pi}\int_B d\,\mathrm{Tr}\left[d\ln(g_1)\wedge(dg_2\,g_2^{-1})\right] \\ =& S^{WZ}_{\mathrm{WZW}}[g_1] + S^{WZ}_{\mathrm{WZW}}[g_2] - \frac{k}{2\pi}\int_\Sigma dz\,\mathrm{Tr}\left[(g_1^{-1}\,\overline{\partial}g_1)\,(\partial g_2\,g_2^{-1})\right] \\ &+ \frac{k}{2\pi}\int_\Sigma dz\,\mathrm{Tr}\left[(g_1^{-1}\,\partial g_1)\,(\overline{\partial}g_2\,g_2^{-1})\right]\,. \end{aligned} \tag{2.2.11}$$

The third step was evaluated using Stokes and local complex coordinates. Adding these terms gives the desired result. □

Let $\Omega_L(z,\bar{z}), \Omega_R(z,\bar{z}) \in C^\infty(\Sigma, G)$ arbitrary. Using (2.2.9) yields

$$\begin{aligned} S_{\mathrm{WZW}}[\Omega_L g \Omega_R] = S_{\mathrm{WZW}}[g] + S_{\mathrm{WZW}}[\Omega_L] + S_{\mathrm{WZW}}[\Omega_R] \\ - \frac{k}{\pi}\int_\Sigma dz\, \mathrm{Tr}\left(g^{-1}\,\bar{\partial} g\, \partial\Omega_R\, \Omega_R^{-1}\right) \\ - \frac{k}{\pi}\int_\Sigma dz\, \mathrm{Tr}\left[\Omega_L^{-1}\,\bar{\partial}\Omega_L\left(\partial g\, g^{-1} + g\,\partial\Omega_R\,\Omega_R^{-1} g^{-1}\right)\right] . \end{aligned} \tag{2.2.12}$$

From this we observe that the WZW model (2.2.2) is invariant under

$$g(z,\bar{z}) \to \Omega_L(z)\, g(z,\bar{z})\, \Omega_R(\bar{z})\,, \tag{2.2.13}$$

that is, it admits a local $G(z) \times G(\bar{z})$ symmetry, thus a gauge symmetry. To compute the associated conserved currents we introduce an infinitesimal parameter $\epsilon(z,\bar{z})$ such that $\Omega_{L/R} \approx \mathbb{1} + \epsilon\, \omega_{L/R}$ with the respective dependencies. Upon an integration by parts this gives

$$\begin{aligned} S_{\mathrm{WZW}}[\Omega_L g \Omega_R] = S_{\mathrm{WZW}}[g] + \mathcal{O}(\epsilon^2) \\ + \frac{k}{\pi}\int_\Sigma dz\, \epsilon(z,\bar{z})\, \mathrm{Tr}\left[\omega_L(z)\,\bar{\partial}(\partial g\, g^{-1}) + \omega_R(z)\,\partial(g^{-1}\,\bar{\partial} g)\right] . \end{aligned} \tag{2.2.14}$$

As ϵ is arbitrary, the conserved current is given by

$$\omega_L(z)\,\bar{\partial}(\partial g\, g^{-1}) + \omega_R(z)\,\partial(g^{-1}\,\bar{\partial} g) = 0\,. \tag{2.2.15}$$

By using the equations of motion (2.2.8) we see that both terms are conserved separately, giving the desired chiral currents. To recapitulate:

Proposition 3. *The WZW model (2.2.2) admits a $G(z) \times G(\bar{z})$ gauge symmetry with associated holomorphic respectively anti-holomorphic currents*

$$J(z) := k\,\partial g\, g^{-1} \quad , \quad \bar{J}(z) := k\, g^{-1}\,\bar{\partial} g\,, \tag{2.2.16}$$

i.e. $\bar{\partial} J = 0$ *and* $\partial \bar{J} = 0$.

Remark 1. As one can see, the currents (2.2.16) could also have been directly extracted separately from the equations of motion (2.2.8) since $\partial(g^{-1}\,\bar{\partial} g) = g^{-1}\bar{\partial}(\partial g\, g^{-1})g$. However, the purpose of the preceding computations was to clarify to which symmetry they correspond.

2.2.3 Current algebra and energy-momentum tensor

So far the discussion was only classical. We would like to elaborate on the CFT of the WZW model by determining the operator product expansion (OPE) of the conserved currents (2.2.16) and deducing the energy-momentum tensor [27]. The currents are associated to the symmetry (2.2.13) which can be written infinitesimally as $\delta_{\omega_L} g = \omega_L g$ and $\delta_{\omega_R} g = \omega_R g$. Varying the holomorphic current $J(z) = J^a(z)\, t^a$ yields[9]

$$\begin{aligned} \delta_{\omega_L} J(z) &= k\left[\partial(\delta_{\omega_L} g)\, g^{-1} + \partial g\, \delta_{\omega_L} g^{-1}\right] \\ &= [\omega_L, J] + k\,\partial\omega_L \\ &= i f_{abc}\, \omega_L^b\, J^c + k\,\partial\omega_L\,. \end{aligned} \tag{2.2.17}$$

On the other hand we can compute $\delta_{\omega_L} J(z)$ by commuting with the conserved charge Q_L associated to (2.2.13) which generates the symmetry transformations. It is given by

$$Q_L^a = \frac{1}{2\pi i}\oint dz\, J^a(z)\, \omega_L(z)\,, \tag{2.2.18}$$

and allows to compute

$$\begin{aligned} \left\langle \delta^a_{\omega_L} J^b(w)\, \ldots \right\rangle &= \left\langle [Q^a, J^b]\, \ldots \right\rangle \\ &= \frac{1}{2\pi i}\oint dz \left\langle [J^a(z)\, \omega_L(z), J^b(w)]\, \ldots \right\rangle \\ &= \frac{1}{2\pi i}\oint_{C(w)} dz\, \omega_L(z) \left\langle R\left[J^a(z)\, J^b(w)\right]\, \ldots \right\rangle . \end{aligned} \tag{2.2.19}$$

In the last step we used radial ordering for the operators J^a and the contour deformation in Figure 2.2 where $C(w)$ denotes the contour around w. "… “ inside the path integral

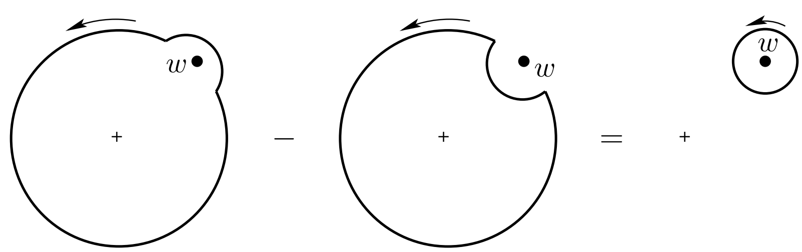

Figure 2.2: The contour deformation between the difference of the two paths.

denote arbitrary other operator insertions. Comparing (2.2.17) and (2.2.19) and using Cauchy's theorem, we derive

Lemma 3. The OPE of the holomorphic currents (2.2.16) of the WZW model (2.2.2) read

$$\left\langle R\left[J^a(z)\, J^b(w)\right]\, \ldots \right\rangle = \left\langle \left[\frac{k\,\delta^{ab}}{(z-w)^2} + i f^{ab}{}_c\, \frac{J^c(w)}{(z-w)} + \text{reg.}\right]\, \ldots \right\rangle , \tag{2.2.20}$$

where reg. denotes the regular, i.e. non-singular terms in the expansion.

[9] In the decomposition $J(z) = J^a(z)\, t^a$, $J^a(z)$ are mere functions/operators.

The discussion of the anti-holomorphic currents $\overline{J}$ is completely analogous. The OPE (2.2.20) will be called a *current algebra*. From now on we will adopt the common notation of *dropping the path integral and the radial ordering*; of course they have to be understood any time products of operators appear. By expanding the holomorphic currents in a Laurent expansion. i.e.

$$J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1}\, j^a_n \iff j^a_n = \oint \frac{dz}{2\pi i}\, z^n\, J^a(z) \tag{2.2.21}$$

we can derive

$$\begin{aligned} [j^a_m, j^b_n] &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i}\, z^m\, w^n\, [J^a(z), J^b(w)] \\ &= \oint_{C(0)} \frac{dw}{2\pi i}\, w^n \oint_{C(w)} \frac{dz}{2\pi i}\, z^m\, J^a(z)\, J^b(w) \\ &= \oint_{C(0)} \frac{dw}{2\pi i}\, w^n \left[k\, m\, w^{m-1}\, \delta^{ab} + w^m\, i f^{ab}{}_c\, J^c(w)\right] \\ &= k\, m\, \delta^{ab}\, \delta_{m,-n} + i f^{ab}{}_c\, j^c_{m+n}\,, \end{aligned} \tag{2.2.22}$$

at which the second step was performed using the contour deformation introduced above for fixed w, the third step using (2.2.20) and the last using Cauchy's theorem and (2.2.21). This algebra is equivalent to (2.2.20).

Definition 3. The algebra

$$[j^a_m, j^b_n] = i f^{ab}{}_c\, j^c_{m+n} + k\, m\, \delta^{ab}\, \delta_{m,-n} \tag{2.2.23}$$

is called the *Kač-Moody* algebra of level k, denoted $\hat{\mathfrak{g}}_k$.

The energy-momentum tensor

Motivated by calculating the classical energy momentum tensor of (2.2.2), the quantum version can be determined by the ansatz

$$T(z) = \gamma \sum_{a=1}^{\dim(\mathfrak{g})} :J^a\, J^a:(z)\,. \tag{2.2.24}$$

One is able to define a CFT via its energy momentum tensor since the form of the TT-OPE,

$$T(z)\, T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{reg.} \tag{2.2.25}$$

is determined merely by conformal invariance in a CFT with central charge c. The route of constructing a CFT by starting with the energy momentum tensor is known as *Sugawara construction*. We determine γ by requiring that J^a is a primary field of conformal weight $h = 1$. One has to be careful in performing the necessary contractions since the WZW model (2.2.2) is in general an *interacting* QFT due to the topological term. This necessitates amplification[10] of the Wick theorem [27].

Lemma 4 (Generalized Wick Theorem)**.** For arbitrary chiral operators $A(z)$, $B(z)$, $C(z)$ we have

$$\overbracket{A(z)\,:\!B}C\!:(w) = \frac{1}{2\pi i}\oint_{C(w)} \frac{dx}{(x-w)}\left[\overbracket{A(z)B}(x)\,C(w) + B(x)\,\overbracket{A(z)C}(w)\right]. \tag{2.2.26}$$

Remark 2. Defining the normal ordered product of two operators in terms of modes by putting all creation operators to the right is equivalent to considering the first regular part in the OPE. Although this is done in the definition it is crucial to observe that the integrand contains, as opposed to the free case full OPE's of the contracted operators; $A(z)B(w) =\!: A(z)B(w)\!: + \underbracket{A(z)B}(y)$.

Now we can compute

$$\begin{aligned}
\overbracket{J^a(z)\,:\!J_b}J^b\!:(w) &= \frac{1}{2\pi i}\oint_{C(w)} \frac{dx}{(x-w)}\left[\overbracket{J^a(z)J_b}(x)\,J^b(w) + J_b(x)\,\overbracket{J^a(z)J^b}(w)\right] \\
&= \frac{1}{2\pi i}\oint_{C(w)} \frac{dx}{(x-w)}\left\{\left[\frac{k\,\delta^{ab}}{(z-x)^2} + i{f^a}_{bc}\frac{J^c(x)}{(z-x)}\right]J^b(w)\right. \\
&\qquad \left. + J^b(x)\left[\frac{k\,\delta^{ab}}{(z-w)^2} + i{f^a}_{bc}\frac{J^c(w)}{(z-w)}\right]\right\} + \text{reg.}
\end{aligned} \tag{2.2.27}$$

where the second step employed (2.2.20). Evaluating the remaining OPE's and using ${f^a}_{bc}\delta^{bc} = 0$, we obtain

$$\begin{aligned}
\overbracket{J^a(z)\,:\!J_b}J^b\!:(w) &= \frac{1}{2\pi i}\oint_{C(w)} \frac{dx}{(x-w)}\left\{\frac{k\,J^a(w)}{(z-x)^2} + \frac{k\,J^a(w)}{(z-w)^2}\right. \\
&\qquad + i{f^a}_{bc}\frac{:\!J^bJ^c\!:(w)}{(z-w)} + \frac{i{f^a}_{bc}}{(z-x)}\left[i{f^{cb}}_d\frac{J^d(w)}{(x-w)}\right. \\
&\qquad \left.\left. + :\!J^cJ^b\!:(w)\right]\right\} + \text{reg.} \\
&= (2k\delta^a_d - {f^a}_{bc}{f^{cb}}_d)\frac{J^d(w)}{(z-w)^2} + i{f^a}_{bc}\frac{:\!J^{(b}J^{c)}\!:(w)}{(z-w)} + \text{reg.} \\
&= 2(k + C_{\mathfrak{g}})\frac{J^a(w)}{(z-w)^2} + \text{reg.}\,.
\end{aligned} \tag{2.2.28}$$

[10] In a free theory the contraction is done with respect to the propagator since it accidentally coincides with the OPE of the primaries. However, in an interacting theory the OPE of primaries in general still contains operators (cf. (2.2.20)). Multiplication with these has also to be given sense by a normal ordering prescription.

For the second step we expanded the $(z-x)$ in the last term to evaluate the integral. The last step used the antisymmetry of the structure constant and substituted the dual Coxeter number $f^a{}_{bc}f^{cbd} = -2C_{\mathfrak{g}}\delta^{ad}$. Thus, upon interchanging z and w we found

$$\begin{aligned} T(z)J^a(w) &= 2\gamma(k+C_{\mathfrak{g}})\frac{J^a(z)}{(z-w)^2} + \text{reg.} \\ &= 2\gamma(k+C_{\mathfrak{g}})\left[\frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)}\right] + \text{reg.}\,, \end{aligned} \tag{2.2.29}$$

i.e. J^a is a primary if we fix $\gamma^{-1} = 2(k+C_{\mathfrak{g}})$. By computing the TT-OPE similarly and comparing with (2.2.25), the central charge of the WZW model can be determined as

$$c = \frac{k\ \dim(\mathfrak{g})}{k+C_{\mathfrak{g}}}\,. \tag{2.2.30}$$

Remark 3. In respect of the following we would like to stress that writing out the normal ordering in the Wick theorem (2.2.26) by the integral is crucial. Examining the calculation (2.2.27) one can observe that upon just contracting the fields and using the OPE without writing out the integral would yield the same result but without the $C_{\mathfrak{g}}$-term. This term is a quadratic contribution from the structure constants. Thus by just contracting $\mathcal{O}(f^2)$ contribution are neglected. This is precisely the regime for the calculations in chapter 4 and hence a pleasing simplification.

2.3 T-duality and Buscher rules

T-duality is one reason for the common statement that string theory sees the spacetime geometry very differently compared to point particles as it identifies a priory distinct theories with different spacetimes. In this section we like to show how background fields change upon performing T-duality transformations and illustrate it at an example which will be important in chapter 4.

2.3.1 The Buscher rules

The rules describing the change of background fields under T-duality transformations were first formulated by Buscher [28]. Our derivation will follow [7]. Consider the sigma model (2.0.4)

$$\begin{aligned} S &= \frac{1}{2\pi\alpha'}\int_\Sigma d^2z\ (G_{\mu\nu}(X) + B_{\mu\nu}(X))\ \partial X^\mu\,\overline{\partial}X^\nu \\ &= \frac{1}{2\pi\alpha'}\int_\Sigma d^2z\, G_{\mu\nu}(X)\,\partial X^\mu\,\overline{\partial}X^\nu + \frac{i}{4\pi\alpha'}\int_B X^*H \end{aligned} \tag{2.3.1}$$

with $H = dB \in \Omega^3(M)$ and $\partial B = \Sigma$ subject to the observations made in the previous section.

Lemma 5. If the metric G admits an isometry, i.e. satisfies $\mathcal{L}_k G_{\mu\nu} = 0$ for a killing vector field $k \in \Gamma(M, TM)$ which is also a symmetry of the action (2.3.1), then there exist local coordinates $Y^\mu = (\theta, Y^a)$, $a \in \{1, \ldots, D-1\}$ such that G and B are independent of θ.

Proof. If k is a killing vector for G, the kinetic term vanishes upon $\delta X^\mu = k^\mu$ and $\mathcal{L}_k G_{\mu\nu} = 0$. For the Wess-Zumino term we employ (2.2.3),

$$\delta \int_B X^* H = \int_B \mathcal{L}_{\delta X} H = \int_B \mathcal{L}_k H \,. \tag{2.3.2}$$

This vanishes if

$$\mathcal{L}_k H = \mathcal{L}_k dB = d\mathcal{L}_k B = 0\,, \tag{2.3.3}$$

where $d \circ \mathcal{L} = \mathcal{L} \circ d$ was used which easily follows from writing $\mathcal{L}$ in terms of the interior and exterior product. Thus the action is invariant under the isometry if $\mathcal{L}_k B = d\omega$ locally for some one-form ω. The Lie derivative can be written locally as

$$\mathcal{L}_k T_{\mu\nu} = k^\sigma \partial_\sigma T_{\mu\nu} + (\partial_\mu k^\sigma) T_{\sigma\nu} + (\partial_\nu k^\sigma) T_{\mu\sigma} \tag{2.3.4}$$

for a rank two tensor T and $k = k^\mu\, \partial/\partial x^\mu$. Upon (2.0.3) $B \to B + dA$ and $d \circ \mathcal{L} = \mathcal{L} \circ d$ we observe that $\omega \to \omega + \mathcal{L}_k A$ is a symmetry. Now we can choose coordinates such that $k = \partial/\partial\theta$; as this is a Killing vector G does not depend on θ. Then (2.3.4) simplifies such that in particular $\mathcal{L}_k A_\mu = \partial_\theta A_\mu$. Since the coefficients A_μ and ω_μ are smooth functions, the Picard-Lindelöf theorem allows for a solution to $\partial_\theta A_\mu = -\omega_\mu$. Thus we can find a gauge with $\omega = 0$, hence $\mathcal{L}_k B = \partial_\theta B = 0$. □

One can obtain an action equivalent to (2.3.1) by "gauging" the isometry, i.e. introducing an action where the isometry appears as a gauge symmetry[11]. The consistency condition is that the gauged action coincides with (2.3.1) when the gauge field is "pure gauge[12]". This can be achieved by the sigma model

$$\begin{aligned} \mathcal{S} = \frac{1}{2\pi\alpha'} \int_\Sigma d^2 z \, \big[& G_{00}\, A\overline{A} + (G_{0a} + B_{0a})\, A\, \overline{\partial} Y^a + (G_{a0} + B_{a0})\, \partial Y^a\, \overline{A} \\ & + (G_{ab} + B_{ab})\, \partial Y^a\, \overline{\partial} Y^b + \tilde{\theta}\, (\partial \overline{A} - \overline{\partial} A) \big]\,, \end{aligned} \tag{2.3.5}$$

with $(Y^0 = \theta, Y^a)$ the adapted coordinates, $\mathcal{A} = A(z)dz + \overline{A}(\overline{z})d\overline{z}$ a gauge field and $\tilde{\theta}(z, \overline{z})$ a Lagrange multiplier. Now we like to check consistency. The equations of motion for $\tilde{\theta}$ can easily be computed as

$$\partial \overline{A} - \overline{\partial} A = 0\,, \tag{2.3.6}$$

[11] For a detailed description one may consult [29].

[12] For a (non-abelian) gauge symmetry we generally have $A \to A' = g^{-1} A g + g^{-1} dg$ and pure gauge would be $A' = d\ln(g)$. In our case of an abelian gauge invariance (2.0.3), pure gauge means that B' is exact, $B' = d\omega$.

which is solved by setting $A = \partial\theta$, $\overline{A} = \overline{\partial}\theta$ (pure gauge). Plugging this back into (2.3.5), renaming $\theta = Y^0$ we indeed recover (2.3.1). We can also compute the equations of motion for the gauge fields and find

$$\begin{aligned} A &= -\frac{G_{a0} + B_{a0}}{G_{00}}\,\partial Y^a + \frac{1}{G_{00}}\,\partial\tilde{\theta} \\ \overline{A} &= -\frac{G_{0a} + B_{0a}}{G_{00}}\,\overline{\partial} Y^a - \frac{1}{G_{00}}\,\overline{\partial}\tilde{\theta}\,. \end{aligned} \tag{2.3.7}$$

If we substitute these back into (2.3.5), some algebraic manipulations, sorting terms by symmetry and denoting $\tilde{\theta} = Y^0$ yields the *dual action*

$$\tilde{S} = \frac{1}{2\pi\alpha'}\int_\Sigma d^2z\,\left(\widetilde{G}_{\mu\nu}(Y) + \widetilde{B}_{\mu\nu}(Y)\right)\,\partial Y^\mu\,\overline{\partial} Y^\nu\,, \tag{2.3.8}$$

where $\widetilde{G}$ and $\widetilde{B}$ are related to the old ones by the

Buscher Rules.

$$\begin{aligned} &\widetilde{G}_{00} = \frac{1}{G_{00}}\ , \qquad \widetilde{G}_{0a} = \frac{B_{0a}}{G_{00}}\ , \qquad \widetilde{G}_{ab} = G_{ab} - \frac{G_{a0}G_{0b} + B_{a0}B_{0b}}{G_{00}} \\ &\widetilde{B}_{0a} = \frac{G_{0a}}{G_{00}}\ , \qquad \widetilde{B}_{ab} = B_{ab} + \frac{G_{a0}B_{0b} + B_{a0}G_{0b}}{G_{00}}\,, \end{aligned} \tag{2.3.9}$$

where the isometry acts in the 0*-direction,*

We remark the following.

- Already in the argument claiming equivalence of (2.3.5) and (2.3.1) we omitted some subtleties. However, Roček and Verlinde [7] showed that the duality is a true symmetry of CFT and thus (2.3.1) and (2.3.8) have to be considered equivalent.
- Considering the easy example of a circle with radius R, the metric in spherical coordinates is given by $ds^2 = (dR)^2 + R^2(d\varphi)^2$ which has φ as isometric direction. Thus applying the Busher rules yields $ds^2 = (dR)^2 + R^{-2}(d\varphi)^2$, reproducing the well-known result that T-duality inverts the radius of the circle.
- The Noether currents associated to the isometry are in general not chiral. In the CFT analysis done in (2.3.1) mentioned above, a second isometry was introduced in order to obtain left- and right-handed (i.e. holomorphic and anti-holomorphic) currents. Along the way of showing equivalence, they revealed the connection between the coordinate θ and $\tilde{\theta}$: $\theta = \theta_L + \theta_R$, $\tilde{\theta}_R = \theta_L - \theta_R$, i.e. *from a CFT perspective T-duality just interchanges $X_R \to -X_R$ along the T-dual direction.*
- Performing a T-duality twice in the same direction gives back the original geometry.
- T-duality is a stringy symmetry which is not apparent in the usual Kaluza-Klein reduction in field theory. Beside the momentum along the compactified directions, the string can also wind around these, quantified by a winding mode. In fact, *T-duality interchanges momentum and winding modes* [21].

2.3.2 Application to $\mathbb{T}^3$

Following [15, 16, 17] we like to illustrate the effect of T-Duality on the sigma model (2.0.4) in a flat, rectangular three-torus $\mathbb{T}^3$ background with local coordinates (x^1, x^2, x^3) whose cycles are parametrized by the Radii R_1, R_2 and R_3. The metric is given by the line element

$$ds^2 = R_1^2\,(dx^1)^2 + R_2^2\,(dx^2)^2 + R_3^2\,(dx^3)^2 \tag{2.3.10}$$

and we will also allow for a Kalb-Ramond field

$$B = c\,x^1\,dx^2 \wedge dx^3 \quad \Longleftrightarrow \quad B_{23} = c\,x^1 \tag{2.3.11}$$

for $c \neq 0$ any constant. In view of the background consistency equations, this is not an admissible string background since (2.1.11) for the flat torus and $H = dB$, i.e. $H_{123} = c$ read

$$H_{\mu\sigma\rho}\,H^{\sigma\rho}{}_{\nu} = (H_{123})^2 = c^2 = 0\,. \tag{2.3.12}$$

However, up to linear order in H this equation is satisfied and thus in this regime we deal with a consistent background which is also with regard to the analysis in chapter 4.

The metric (2.3.10) has three isometries; one in each direction of the cycles and we are free to choose any of them to perform a T-duality by applying the Buscher rules (2.3.9). The discussion is simplified by considering $\mathbb{T}^3$ as a two-torus $\mathbb{T}^2$ in the (x^2, x^3)-direction fibered over a circle S^1 in the x^1 direction. This background will be generally referred

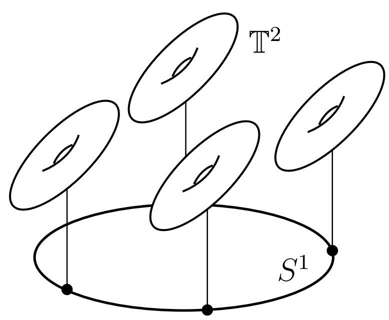

Figure 2.3: A cartoon of a two-torus $\mathbb{T}^2$ fibered over a circle S^1.

to as *H-flux* background. Now we can apply the Buscher rules and study the resulting geometries.

The twisted torus

Utilizing (2.3.9) we can perform a first T-duality in the, say, x^3 direction to obtain

$$\begin{aligned} ds^2 &= R_1^2\,(dx^1)^2 + R_2^2\,(dx^2)^2 + \frac{1}{R_3^2}\left(dx^3 + c\,x^1\,dx^2\right)^2 \\ B &= 0\,, \end{aligned} \tag{2.3.13}$$

where we can observe that the radius of the third cycle has been inverted and the x^1-direction is not isometric anymore. Since x^1 is the direction along the base space S^1, in order for the metric to be well-defined the line element must be invariant under $x^1 \to x^1 + 2\pi R_1$, i.e. has to respect the periodicity. This can be restored by imposing the identification

$$(x^1, x^2, x^3) \sim (x^1 + 2\pi\, R_1, x^2, x^3 - 2\pi\, R_1\, c\, x^2) \tag{2.3.14}$$

and defines the so-called *twisted torus*. In general we will refer to this background as the *geometric* or *ω-flux*. This can be motivated as follows. If we introduce a dual basis of globally defined one forms (Vielbeine)

$$\eta^1 = dx^1 \ , \qquad \eta^2 = dx^2 \ , \qquad \eta^3 = dx^3 + c\, x^1\, dx^2 \tag{2.3.15}$$

such that the metric (2.3.13) is diagonal, Cartan's structure equation for a torsion-free connection reads

$$d\eta^a = \eta^b \wedge \omega^a{}_b \ , \tag{2.3.16}$$

where $\omega^a{}_b$ is the spin connection[13]. Using the basis defined above the only non-vanishing component of $\omega^a{}_b$ turns out to be

$$\omega_{12}{}^3 = -c\,. \tag{2.3.17}$$

As in the previous case, we will name this background after the quantity related to c, which justifies the above choice.

The T-fold

The second direction x^2 of $\mathbb{T}^2$ is still isometric in (2.3.13), allowing for a T-duality which gives

$$\begin{aligned} ds^2 &= R_1^2\,(dx^1)^2 + \frac{1}{R_2^2\,R_3^2 + c^2\,(x^1)^2}\left(R_3^2\,(dx^2)^2 + R_2^2\,(dx^3)^2\right) \\ B_{23} &= -\frac{c\,x^1}{R_2^2\,R_3^2 + c^2\,(x^1)^2}\,. \end{aligned} \tag{2.3.18}$$

Although the metric and the B-field are well-defined locally, i.e. with respect to the represented patch, one is not able to "glue" the patches together in a consistent manner. The reason is that the transition functions between local trivializations of the bundle $\mathbb{T}^2 \to S^1$ mix the B-field with the metric. This makes sense as in section 2.1 we discovered that the H-field can be considered a geometric datum of the manifold. Including this mixing in the transition functions lead to the notion of *T-folds* [18].

In this case the quantity related to c is denoted Q_1^{23} and can be computed in the context of *generalized geometry*[14]. Thus we call this background *non-geometric* or *Q-flux* background.

---

[13] The spin connection enters here since introducing vielbeine, roughly speaking, maps the tangent bundle to the vector bundle associated to the frame bundle whose natural connection is the spin connection.

[14] For this particular example, see e.g. [30] and references therein.

The R-flux

Finally we can also consider a T-duality along the base direction x^1 which is not captured by the Buscher rules as (2.3.18) doesn't admit an isometry in this direction. In [16, 17] it is argued that although the background obtained by performing another T-duality seems to elude a geometric description even locally, it has to be included in a background independent formulation of string theory. We formally characterize this background by a new type of flux, denoted by $R^{123} = N$.

To recapitulate, we have described a chain of T-dualities

$$H_{123} \quad \overset{T_3}{\longleftrightarrow} \quad \omega_{12}{}^3 \quad \overset{T_2}{\longleftrightarrow} \quad Q_1{}^{23} \quad \overset{T_1}{\longleftrightarrow} \quad R^{123} \tag{2.3.19}$$

in which the first and the last background will be of the main interest in the following chapters.

Chapter 3

Open strings and noncommutative geometry

This chapter is devoted to the emergence of noncommutative geometry on D-branes and we will mainly follow [8, 10, 31].

3.1 Open strings on D-branes with B-field

The starting point will be the sigma model of an open string on a Dp-brane with a constant B-field in a flat background. Due to (2.1.11) this is an admissible background since $H = 0$. We will describe this configuration classically in the following. Because it simplifies the discussion as we want to consider this problem in terms of mode expansions, in this chapter we want the worldsheet to have *Lorentzian* signature and choose conformal gauge $h_{\alpha\beta} = \mathrm{diag}(-1, 1)$. The action is given by

$$S_o = \frac{1}{4\pi\alpha'} \int_{\Sigma_o} d^2\sigma \, \big(h^{\alpha\beta}\, \eta_{\mu\nu} + \epsilon^{\alpha\beta}\, \mathcal{F}_{ij}(X)\big)\, \partial_\alpha X^\mu \, \partial_\beta X^\nu \, , \tag{3.1.1}$$

where we chose a flat background $G_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$ and

$$\mathcal{F} = B + dA \, . \tag{3.1.2}$$

A is a $U(1)$ gauge field on the spacetime which has to be added to the action in order for (2.0.3) to hold since the worldsheet is bounded now: $\partial\Sigma_o \neq \emptyset$. The gauge symmetry (2.0.3) is modified to

$$B \to B + d\omega \;\; , \qquad A \to A - \omega \, . \tag{3.1.3}$$

Moreover, while μ, ν denote the coordinates on the whole target space, $i, j \in \{0, \ldots, p\}$ denote the coordinates along the Dp-brane. Restricting B to the brane can always be realized by the gauge transformations as the interior of the worldsheet is closed.

For later purpose we are interested in the equations of motion of (3.1.1). By using the antisymmetry of $\mathcal{F}$, varying S_o yields

$$\delta S_o \sim \int_{\Sigma_o} d^2\sigma \Big[-\partial_\alpha \partial^\alpha X^\mu \, \delta X_\mu + \frac{1}{2}\epsilon^{\alpha\beta} \, (d\mathcal{F})_{ijk} \partial_\alpha X^i \, \partial_\beta X^j \, \delta X^k \\ + \partial_\alpha (\delta X_\mu \, \partial^\alpha X^\mu) + \epsilon^{\alpha\beta} \, \mathcal{F}_{ij} \, \partial_\alpha (\delta X^i \, \partial_\beta X^j) \Big] \, . \tag{3.1.4}$$

The second term in the first line vanishes and the second line collects the boundary terms which vanish by imposing the respective boundary conditions on the brane and transverse to it. Thus the equation of motions are

$$\partial_\alpha \partial^\alpha X^\mu = 0 \tag{3.1.5}$$

subject to the boundary conditions

$$\begin{aligned} \left(\partial_\sigma X^i - \mathcal{F}^i{}_j \partial_\tau X^j\right)\Big|_{\sigma=0,\pi} &= 0 \ , \qquad & i,j &= 0,\dots,p \\ X^a\Big|_{\sigma=0,\pi} &= x_0^a \ , \qquad & a &= p+1,\dots,D \, , \end{aligned} \tag{3.1.6}$$

where we parametrized the worldsheet by $\sigma \in [0,\pi]$ and a denotes the directions transverse to the brane. Thus we are able to solve (3.1.5) by the usual mode expansion, which reads

$$\begin{aligned} X^i(\tau,\sigma) &= x_0^i + \left(p_0^i \tau - p_0^j \, \mathcal{F}_j{}^i \sigma\right) + \sum_{n\neq 0} \frac{e^{-in\tau}}{n} \left(i\alpha_n^i \cos(n\sigma) - \alpha_n^j \mathcal{F}_j{}^i \sin(n\sigma)\right) \\ X^a(\tau,\sigma) &= x_0^a + b^a \sigma + \sum_{n\neq 0} \frac{e^{-in\tau}}{n} \, \alpha_n^a \sin(n\sigma) \end{aligned} \tag{3.1.7}$$

with respect to the boundary conditions (3.1.6). Now we want to quantize this system.

3.1.1 Emergence of noncommutative geometry

The aim is to derive the symplectic form of the spacetime which defines the Poisson structure; then the commutation relations can be read-off. Determining the phase space of a system with constraints, which here are given by the boundary conditions (3.1.6) can be done by the quantization procedure formulated by P. Dirac [32]. However, Chu and Ho proposed a rather efficient method suitable for this problem [8] which was shown to be equivalent to Dirac's procedure in [33].

First we determine the canonical momenta for (3.1.1) as

$$\begin{aligned} 2\pi\alpha' \, P^i(\tau,\sigma) &= \partial_\tau X^i + \mathcal{F}^i{}_j \partial_\sigma X^j = \Big[p_0^k + \sum_{n\neq 0} e^{-in\tau} \, \alpha_n^k \, \cos(n\sigma)\Big] M_k{}^i \\ 2\pi\alpha' \, P^a(\tau,\sigma) &= \partial_\tau X^a = -i \sum_{n\neq 0} e^{-in\tau} \, \alpha_n^a \, \sin(n\sigma) \end{aligned} \tag{3.1.8}$$

with $M_{ij} = \eta_{ij} - \mathcal{F}_i{}^k \mathcal{F}_{kj}$. For a symplectic form Chu and Ho propose to use

$$\begin{aligned}\omega &:= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} d\tau \int_0^\pi d\sigma\, dP_\mu \wedge dX^\mu \\ &= \frac{1}{2\alpha'} \left[M_{ij}\, dp_0^i \wedge (dx_0^j + \frac{\pi}{2} \mathcal{F}^j{}_k dp_0^k) - i \sum_{n>0} \frac{1}{n} (M_{ij} d\alpha_n^i \wedge d\alpha_{-n}^i + d\alpha_n^a \wedge d\alpha_{-n}^a) \right]\end{aligned} \tag{3.1.9}$$

where (3.1.7) and (3.1.8) were already plugged-in. The novelty of (3.1.9) is that the fields X and P were not used to determine their commutation relations directly but rather the modes[1] and to take a time average in order for the result to be independent of τ. This is necessary for this to be a bona-fide spacetime object.

The Poisson structure is determined by the inverse of the components of the symplectic form (3.1.9). The block related to x_0 and p_0 reads

$$\omega = \frac{1}{2\alpha'} \begin{pmatrix} 0 & -\frac{1}{2}M \\ \frac{1}{2}M & \frac{\pi}{2} M\mathcal{F} \end{pmatrix} \qquad \Longrightarrow \qquad \omega^{-1} = \begin{pmatrix} 4\pi\alpha' M^{-1}\mathcal{F} & 4\alpha' M^{-1} \\ -4\alpha' M^{-1} & 0 \end{pmatrix} . \tag{3.1.10}$$

The α-block is obvious and thus the commutation relations, as obtained by multiplying with i ($\hbar \equiv 1$), read

$$\begin{aligned} [x_0^i, x_0^j] &= 4i\pi\alpha' \, (M^{-1}\mathcal{F})^{ij} \\ [x_0^i, p_0^j] &= 4i\alpha' \, (M^{-1})^{ij} \\ [\alpha_m^i, \alpha_n^j] &= 2\alpha' m \, (M^{-1})^{ij} \, \delta_{m+n} \\ [\alpha_m^a, \alpha_n^b] &= 2\alpha' m \, \delta^{ab} \, \delta_{m+n} \end{aligned} \tag{3.1.11}$$

and the rest zero. Thus on the brane the commutation relations look unusual. Since we like to find noncommutative geometry, we are particularly interested in the commutator of the spacetime coordinates X. Using the mode expansion (3.1.7) and the above relations yield an equal time commutator

$$\left[X^i(\tau,\sigma), X^j(\tau,\sigma')\right] = -4i\alpha' \, (M^{-1}\mathcal{F})^{ij} \left[(\sigma + \sigma' - \pi) + \sum_{n\neq 0} \frac{1}{n} \sin n(\sigma+\sigma') \right] . \tag{3.1.12}$$

The infinite sum can be evaluated to $\pi - \sigma + \sigma'$ if $(\sigma + \sigma') \in (0, 2\pi)$; for σ and σ' both 0 or π the sum vanishes. Thus, the coordinates commute in the interior of the worldsheet since the term in parentheses cancels. However, at the boundary we find

$$\left[X^i(\tau,\sigma), X^j(\tau,\sigma')\right]\Big|_{\sigma=\sigma'=0/\pi} = \pm\, 4\pi i\alpha' \, (M^{-1}\mathcal{F})^{ij} , \tag{3.1.13}$$

that is, *the coordinates on each of the D-branes are noncommutative.* As also required by consistency we observe that the right-hand-side of (3.1.13) is antisymmetric in i and j.

[1] Thus this is somehow an *extended* phase space with infinitely many coordinates.

3.2 The Moyal star-product

We would like to define a product capturing the noncommutativity of the coordinates X (3.1.13) on the D-branes which we will denote as submanifold $N \subset M$ in the following. When the coordinates of a manifold are noncommutative, of course all functions $f \in C^\infty(N, \mathbb{R})$ are and thus we seek a product on the algebra of functions on the spacetime. For simplicity we will rewrite (3.1.13) as

$$\left[x^i, x^j\right] = i\,\theta^{ij}\,, \tag{3.2.1}$$

where we focus on one of the branes and θ^{ij} is antisymmetric. We will consider the following product [34],

Definition 4. Let $f, g \in C^\infty(N, \mathbb{R})$. Then

$$(f \star g)(x) := \exp\left(\frac{i}{2}\,\theta^{ij}\,\frac{\partial}{\partial x^i}\,\frac{\partial}{\partial y^i}\right) f(x)\,g(y)\bigg|_{x=y} \tag{3.2.2}$$

is called the *Moyal star-product*.

This allows for the following observations

- For the commutator of two coordinates (3.2.2) yields upon expansion of the exponential

$$\begin{aligned}\left[x^i, x^j\right] &= x^i \star x^j - x^j \star x^i \\ &= \left(x^i\,x^j + \tfrac{i}{2}\,\theta^{ij}\right) - \left(x^j\,x^i + \tfrac{i}{2}\,\theta^{ji}\right) \\ &= i\,\theta^{ij}\,.\end{aligned} \tag{3.2.3}$$

  Hence the definition captures (3.2.1).

- More generally, for two functions f, g the definition can be expanded as

$$\begin{aligned}(f \star g)(x) = f(x)\,g(x) + \tfrac{i}{2}\,\theta^{ij}\,\partial_i f(x)\,\partial_j g(x) - \tfrac{1}{8}\,\theta^{ij}\,\theta^{kl}\,\partial_i\partial_k f(x)\,\partial_j\partial_l g(x) \\ - \tfrac{1}{8}\,\theta^{ij}\,\partial_j\theta^{kl}\left(\partial_i\partial_k f(x)\,\partial_l g(x) - \partial_k f(x)\,\partial_i\partial_l g(x)\right) + \mathcal{O}(\theta^3)\,.\end{aligned} \tag{3.2.4}$$

- Using the previous expansion one can show that the *associator* gives in quadratic order in θ

$$\begin{aligned}(f \star g) \star h - f \star (g \star h) &\sim \left(\theta^{il}\,\partial_l\theta^{jk} + \theta^{jl}\,\partial_l\theta^{ki} + \theta^{kl}\,\partial_l\theta^{ij}\right)\partial_i f\,\partial_j g\,\partial_k h \\ &\sim \theta^{ia}\,\theta^{jb}\,\theta^{kc}\,\left[d\left(\theta^{-1}\right)\right]_{abc}\partial_i f\,\partial_j g\,\partial_k h\,.\end{aligned} \tag{3.2.5}$$

  In the second line we assumed that θ is invertible which is motivated by (3.1.13). Thus *the Moyal product* (3.2.2) *is associative if* θ^{-1} *is closed*, which is indeed the case in the previous section. Otherwise the product is nonassociative as extensively discussed in [31].

3.3 The structure of CFT correlators

In this section we want to discuss how the noncommutativity manifests itself in the CFT of the open string (3.1.1) as shown in [10]. We will also encounter a different way of obtaining (3.1.13) and (3.2.2).

The setup will be the sigma model (3.1.1) on an Euclidean worldsheet $h^{\alpha\beta} = \mathrm{diag}(1,1)$ with a constant B-field and now also with a constant A, i.e. the corresponding term is absent; $\mathcal{F} = B$. In complex coordinates introduced in chapter 2 the action (3.1.1) restricted to the Dp-brane reads

$$S_o = \frac{1}{2\pi\alpha'} \int_{\Sigma_o} d^2z \, \left(\eta_{ij}(X) + B_{ij}(X)\right) \partial X^i \,\overline{\partial} X^j \,. \tag{3.3.1}$$

We will also assume Σ_o to have the topology of a disc which we considered mapped to $\mathbb{H}^-$ as above. The integrand can be seen as kinetic term and the propagator, i.e. the fundamental solution to (3.1.5) subject to the boundary conditions (3.1.6) can be determined using the method of image charges. We will focus on open string vertex operators which are inserted on the boundary of $\mathbb{H}^-$. Then the propagator for $\mathbb{H}^-$ restricted to $\mathbb{R}$ reads [10]

$$\mathfrak{G}^{ij}(t,t') := \left\langle X^i(t)\, X^j(t')\right\rangle = -\alpha'\, g^{ij}\, \ln(t-t')^2 + \frac{i}{2}\,\theta^{ij}\,\epsilon(t-t') \tag{3.3.2}$$

with $t, t' \in \mathbb{R}$ and $\epsilon(t)$ is ± 1 for $t \gtrless 0$. g and θ are related to the metric and Kalb-Ramond field by [31]

$$g^{-1} + \frac{1}{2\pi\alpha'}\theta := (\eta + B)^{-1} \,. \tag{3.3.3}$$

Remark 4. In particular, θ is the antisymmetric part of the right-hand side of (3.3.3) which coincides with the right-hand side of (3.1.13). Indeed, we can also extract the commutation relations from the propagator. This can be done similar to the computation (2.2.19). As the product of operators is just well-defined radially ordered, the equal-time, equal-position commutator reads

$$\begin{aligned}\left\langle R\left\{\left[X^i(t), X^j(t)\right]\right\}\right\rangle &= \lim_{\delta\to 0}\left\langle X^i(t)\, X^j(t-\delta) - X^i(t)\, X^j(t+\delta)\right\rangle \\ &= i\,\theta^{ij}\left\langle \mathbb{1}\right\rangle\end{aligned} \tag{3.3.4}$$

with the operator product evaluated using (3.3.2). For this observation the behavior of the ϵ is crucial since it is sensitive to the ordering of the operators; in particular, it is not continuous at 0 since it features a jump.

Now we would like to compute CFT correlators in the theory. A generic vertex operator in the boundary CFT underlying the open string is of the form

$$P\left[(\partial+\overline{\partial})X, (\partial+\overline{\partial})^2 X, \ldots\right](t)\, \exp(ip\cdot X(t)) \tag{3.3.5}$$

with P an arbitrary polynomial of the boundary derivatives[2] of X and p the momentum of the state[3]. In particular, the exponential is the operator corresponding to the tachyon. Our aim is to study the general N-point correlator

$$C^N_{g,\theta} := \left\langle \prod_{n=1}^{N} :P_n[\dots](t_n)\, e^{ip^n\cdot X(t_n)}: \right\rangle_{g,\theta} \tag{3.3.6}$$

of N arbitrary vertex operators (3.3.5) inserted at boundary points t_n. The subscript g,θ indicates that the path integral is with respect to the action (3.3.1). Since we are dealing with a free theory, the contractions are made with respect to the propagator (3.3.2) as it coincides with the OPE. The combinatorics necessary especially for contracting the exponential functions is generally captured in the functional operator [21]

$$:\mathcal{F}(X)::\mathcal{G}(X):= \exp\left[\int d^2z\, d^2w\, \mathfrak{G}^{ij}(z,w)\,\frac{\delta}{\delta X^i_F(z,\overline{z})}\,\frac{\delta}{\delta X^j_G(w,\overline{w})}\right] :\mathcal{F}\,\mathcal{G}:\,, \tag{3.3.7}$$

where $\mathcal{F}$ and $\mathcal{G}$ are operators composed of X and the subscripts in the functional derivatives indicates on which of the operators they act. In our case we restrict to real values and can compute

$$\begin{aligned}\prod_{n=1}^{N} :e^{ip^n\cdot X(t_n)}: &= \exp\left[\int dt\, dt'\, \mathfrak{G}^{ij}(t,t')\,\frac{\delta}{\delta X^i_F(t)}\,\frac{\delta}{\delta X^j_G(t')}\right] :\prod_{n=1}^{N} e^{ip^n\cdot X(t_n)}: \\ &= e^{-\sum_{n<m}\mathfrak{G}^{ij}(t_n,t_m)\,p^n_i\,p^m_j} :\prod_{n=1}^{N} e^{ip^n\cdot X(t_n)}: \\ &= e^{-\sum_{n<m}\left(-\alpha' g^{ij}\ln(t_n-t_m)^2+\frac{i}{2}\theta^{ij}\epsilon(t_n-t_m)\right)p^n_i p^m_j} :\prod_{n=1}^{N} e^{ip^n\cdot X(t_n)}: \\ &= e^{-\frac{i}{2}\sum_{n<m}\theta^{ij}\,p^n_i\,p^m_j\,\epsilon(t_n-t_m)}\left[\prod_{n<m}(t_n-t_m)^{2\alpha' g^{ij}p^n_i p^m_j}\right] :e^{\sum_{n=1}^{N} ip^n\cdot X(t_n)}:\end{aligned} \tag{3.3.8}$$

The ordered sum respectively product ensures that we don't count any contributions twice. With this we are also able to compute (3.3.6). However, since θ is constant the contributions from contractions including θ solely come from contracting the tachyon vertex operators. Thus the correlator can be written very generally as

$$C^N_{g,\theta} = e^{-\frac{i}{2}\sum_{n<m}\theta^{ij}\,p^n_i\,p^m_j\,\epsilon(t_n-t_m)}\, C^N_{g,\theta=0}\,. \tag{3.3.9}$$

That is, the θ dependence can be completely decoupled from the rest of the correlator. We will make the following remarks.

[2]Parametrized by (τ,σ), the boundary $\partial\Sigma_o$ is along τ. In complex coordinates the τ-derivative is given by $\partial+\overline{\partial}$.

[3]Which can be verified via the *operator-state correspondence*.

- Independent of the form of the polynomial P, after we have performed all possible contractions we will be in any case left with something proportional to

$$\left\langle : e^{\sum_{n=1}^{N} ip^n \cdot X(t_n)} : \right\rangle = \delta\left(\sum_{n=1}^{N} p^n\right) , \tag{3.3.10}$$

  which can be directly seen from (3.3.7). When the momenta do not sum-up to zero we are left with a one-pint correlator which vanishes very generally due to conformal invariance. This then yields the δ-distribution and hence implies momentum conservation.

- The conformal Killing group of the disc is the Möbius group with real coefficients $\mathrm{PSL}(2, \mathbb{R})$ and thus (3.3.9) has to be invariant under the action of the group which amounts to cyclically permute the operators inserted. But by using momentum conservation we can see that the θ factor is indeed invariant under cyclic permutations as the remaining part is very generally. Hence (3.3.9) is indeed a genuine CFT correlator.

- Assuming that the operators were inserted at $t_n > t_m \ \forall n < m$ and identifying $p_n = i\partial_{X^n}$ the prefactor of (3.3.9) can be seen to coincide with the Moyal product (3.2.2) for $N = 2$. More generally, this can be considered a definition of an N-ary product

$$(f_1 \star f_2 \star \cdots \star f_N)(x) := \exp\left(\frac{i}{2}\sum_{n<m}^{N} \theta^{ij} \frac{\partial}{\partial x_n^i}\frac{\partial}{\partial x_m^j}\right) f_1(x_1)\dots f_N(x_N)\bigg|_{x_n=x} . \tag{3.3.11}$$

  Since here θ is constant and thus the Moyal product associative this definition is indeed just the successive application of the binary product. Applied to $f_i(x) = \exp(ip^i \cdot x)$ it also gives back the phase.

To summarize, we showed how noncommutative geometry arises in open bosonic string theory. Moreover, from a relative phase apparent by comparing the CFT correlation functions for the theory with B-field we were able to derive the well-known noncommutative Moyal star-product of the algebra of functions. The following main part of the thesis is to some extend dedicated to the detection of such a phase in a closed string analogue of the theory considered here.

Chapter 4

Closed strings and nonassociative geometry

In chapter 3 we have seen how noncommutative geometry arises in open bosonic string theory in the presence of a constant B-field. In this chapter we want to proceed analogously for the closed string. However, a constant B-field appears as a boundary term and hence can be "gauged away" since the worldsheet is closed. Thus the simplest configuration we can consider is a constant H-flux. This chapter mainly follows [20].

4.1 Closed strings with constant H-flux at $\mathcal{O}(H)$

We consider the action (2.0.4) with a constant H-flux. That is, the B-field can be expanded in normal coordinates as [31]

$$B_{\mu\nu}(X) = B_{\mu\nu} + \frac{1}{3}\, H_{\mu\nu\sigma} X^{\sigma} + \dots \tag{4.1.1}$$

with $H_{\mu\nu\sigma}$ constant and totally anti-symmetric. We will neglect higher orders and thus $H = dB$ as in (2.1.5); the constant B-term does not contribute by Stokes' theorem. Moreover we will assume three directions of the target space M, say with local coordinates (X^a), $a \in \{1, 2, 3\}$, to be compactified on a flat, rectangular three-torus $\mathbb{T}^3$ with metric

$$ds^2 = (dX^1)^2 + (dX^2)^2 + (dX^3)^2 \qquad \Longrightarrow \qquad G_{ab} = \delta_{ab} \tag{4.1.2}$$

and $X^a \sim X^a + 2\pi$. We also assume H to be nontrivial just on the torus, i.e. $H_{\mu\nu\sigma} = H_{abc}$. As already mentioned in subsection 2.3.2 this is not an admissible string background due to (2.1.11). Hence we will *just work up to linear order in* H.

In conformal gauge the action restricted to the torus reads

$$\mathcal{S} = \frac{1}{2\pi\alpha'} \int_{\Sigma} d^2 z \left(\delta_{ab} + \frac{1}{3}\, H_{abc} X^c \right) \partial X^a\, \overline{\partial} X^b \tag{4.1.3}$$

which is a bona-fide CFT up to linear order in H. The equations of motions for (4.1.3) are readily found to be

$$\partial\overline{\partial} X^a = \frac{1}{2}\, H^a{}_{bc}\, \partial X^b\, \overline{\partial} X^c\,. \tag{4.1.4}$$

Since we restricted to $\mathcal{O}(H)$ the fields on the right-hand side of (4.1.4) are given by the free field. These are denoted $X_0(z,\overline{z}) = X_{0,L}(z) + X_{0,R}(\overline{z})$ as this is the most general solution to the free wave equation (4.1.4) for $H=0$. A classical solution to (4.1.4) up to linear order in H can be found in terms of the free field as

$$X^a(z,\overline{z}) = X_0^a(z,\overline{z}) + \tfrac{1}{2}\, H^a{}_{bc}\, X^b_{0,L}(z)\, X^c_{0,R}(\overline{z})\,. \tag{4.1.5}$$

From now on we will work up to linear order in H, hence omitting "$\dots + \mathcal{O}(H^2)$" in all the calculations.

4.1.1 Canonical and physical momentum

In a free theory the canonical and physical momentum coincide. However, in the interacting theory (4.1.3) the situation is more subtle and we will explain the difference in the following.

The canonical momentum can be computed from (4.1.3) by writing it in (τ,σ)-coordinates (cf. (2.0.4)). We find

$$\begin{aligned}\pi^a(\tau,\sigma) &= \frac{1}{2\pi\alpha'}\left(\partial_\tau X^a(\tau,\sigma) + \tfrac{i}{3}\, H^a{}_{bc}\, X^c(\tau,\sigma)\, \partial_\sigma X^b(\tau,\sigma)\right)\\ &= \frac{1}{2\pi\alpha'}\left(\partial X^a(z,\overline{z}) - \tfrac{1}{3}\, H^a{}_{bc}\, \partial X^b(z,\overline{z})\, X^c(z,\overline{z})\right)\\ &\quad + \frac{1}{2\pi\alpha'}\left(\overline{\partial} X^a(z,\overline{z}) - \tfrac{1}{3}\, H^a{}_{bc}\, X^b(z,\overline{z})\, \overline{\partial} X^c(z,\overline{z})\right)\,,\end{aligned} \tag{4.1.6}$$

where we employed $\partial_\tau = \partial + \overline{\partial}$ and $\partial_\sigma = i(\partial - \overline{\partial})$ for the last line. This has the usual term involving the "magnetic" field H. In an attempt to canonically quantize this system, (X,π) would be the right choice of canonical coordinates to describe the phase space of the system. However, the *physical* momentum is the quantity related to a force F via $\dot{p} = F$ which means "$p = mv$"[1]. Thus the physical momentum differs from the canonical momentum and is given by

$$p^a(z,\overline{z}) = \frac{1}{2\pi\alpha'}\, \partial_\tau X^a(\tau,\sigma) = \frac{i}{2\pi\alpha'}\left(\partial X^a(z,\overline{z}) + \overline{\partial} X^a(z,\overline{z})\right)\,. \tag{4.1.7}$$

To determine the total momentum it is useful to look at the mode expansion. In the free theory it can be found to read [21]

$$X_0^a(z,\overline{z}) = (x_L^a + x_R^a) - i\frac{\alpha'}{2}\left(k_L^a\, \ln(z) + k_R^a\, \ln(\overline{z})\right) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\left(\frac{\alpha_n^a}{z^n} + \frac{\widetilde{\alpha}_n^a}{\overline{z}^n}\right) \tag{4.1.8}$$

with

$$k_L^a = p^a + \tfrac{1}{\alpha'}\, w^a \quad , \quad k_R^a = p^a - \tfrac{1}{\alpha'}\, w^a\,. \tag{4.1.9}$$

[1] The suitable classical example in our case is the Lorentz force $\mathbf{F}_L = \mathbf{v}\times\mathbf{B}$ for a particle with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$.

Here p^a denotes the (Kaluza-Klein) momentum and w^a the winding along the compact directions. Using the solution (4.1.5) yields

$$\begin{aligned}
\partial X^a(z,\bar{z}) &= -i\frac{\alpha'}{2}\, k_L^a\, \frac{1}{z} + \frac{1}{2} H^a{}_{bc} \left(\frac{\alpha'}{2i}\, k_L^b\, x_R^c\, \frac{1}{z} - \frac{(\alpha')^2}{4}\, k_L^b\, k_R^c\, \frac{\ln(\bar{z})}{z} \right) \\
&\quad + \{\mathcal{O}(z^n),\, n \neq -1\} \\
\bar{\partial} X^a(z,\bar{z}) &= -i\frac{\alpha'}{2}\, k_R^a\, \frac{1}{\bar{z}} + \frac{1}{2} H^a{}_{bc} \left(\frac{\alpha'}{2i}\, x_L^b\, k_R^c\, \frac{1}{\bar{z}} - \frac{(\alpha')^2}{4}\, k_L^b\, k_R^c\, \frac{\ln(z)}{\bar{z}} \right) \\
&\quad + \{\mathcal{O}(\bar{z}^n),\, n \neq -1\}\ .
\end{aligned} \tag{4.1.10}$$

We have given the mode expansion just up to linear singularities since the total momentum is obtained by integrating out the space-, i.e. σ-dependence; this amounts to compute

$$\begin{aligned}
P^a &= \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma\, \partial_\tau X^a(\tau,\sigma) \\
&= \frac{i}{\alpha'} \oint_{C(0)} \frac{dz}{2\pi i}\, \partial X^a(z,\bar{z}) - \frac{i}{\alpha'} \oint_{C(0)} \frac{d\bar{z}}{2\pi i}\, \bar{\partial} X^a(z,\bar{z}) \\
&= \frac{1}{2}\, (k_L^a + k_R^a) = p^a
\end{aligned} \tag{4.1.11}$$

where the contours are counter-clock-wise and Cauchy's theorem was employed. Moreover, the $k_L k_R$-terms cancel and we also exploited that Hx corresponds to a constant B-field term which can be neglected since it appears as a total derivative in the action. Hence (4.1.7) still has the same total momentum as the free theory. Moreover, in the quantum theory $P^a = P_L^a + P_R^a$ will serve as the proper momentum operator.

If we would have changed the sign between the two integrals in the last computation we would obtain the total winding in the free theory. However, here the $k_L k_R$-terms add-up due to the changed relative sign. Thus, in order for the classical states to still have the same winding as the free theory, we have to demand

$$H^a{}_{bc}\, k_L^b\, k_R^c = -\tfrac{2}{\alpha'}\, H^a{}_{bc}\, p^b\, w^c = 0 \tag{4.1.12}$$

by the antisymmetry of H. At this point such a constraint seems rather artificial but we will actually encounter the same restriction later when considering the quantum theory.

4.2 The H-CFT

In the previous chapter 3 we considered a free theory whose CFT description is well-known. However, (4.1.3) is an interacting theory and requires a new CFT framework in order to analyze the structure of the quantum theory in the spirit of section 3.3. This will be achieved in the following and called CFT_H.

4.2.1 A holomorphic current

To utilize techniques of CFT we need to find proper conformal fields of our theory which are apparently not the bosonic fields X as their behavior under conformal transformations is not well-defined. We have already perceived that the action (4.1.3) can be considered as a WZW model (2.2.2). Thus we can find chiral currents from those symmetries of the theory which are generally not directly apparent in (2.0.4).

Rather surprisingly (4.1.3) is invariant under spacetime translations

$$X^a(z,\overline{z}) \to X^a(z,\overline{z}) + a^a \tag{4.2.1}$$

which in the following will be verified and the associated Noether current determined. The variation of $\mathcal{S}$ subject to $\delta X^a = a^a$ reads

$$\begin{aligned}\delta\mathcal{S} &= \frac{1}{2\pi\alpha'}\int_\Sigma d^2z\,\Big[\partial a^a\,\overline{\partial}X_a + \partial X_a\,\overline{\partial}a^a \\ &\qquad + \frac{1}{3}H_{abc}\left(a^a\,\partial X_b\,\overline{\partial}X_c - \partial a^a\,X^b\,\overline{\partial}X^c - \overline{\partial}a^a\,\partial X^b\,X^c\right)\Big] \\ &= \frac{1}{2\pi\alpha'}\int_\Sigma d^2z\,\Big[\partial a^a\left(\overline{\partial}X_a - \frac{1}{2}H_{abc}\,X^b\,\overline{\partial}X^c\right) \\ &\qquad + \overline{\partial}a^a\left(\partial X_a - \frac{1}{2}H_{abc}\,\partial X^b\,X^c\right)\Big].\end{aligned} \tag{4.2.2}$$

In the second step we split the the first term in the second line in halves and performed Stokes' theorem for z respectively $\overline{z}$. Thus for a constant we showed that (4.1.3) is indeed invariant under spacetime translations. Upon another application of Stokes' theorem we also find the conservation law

$$\begin{aligned}0 &= \partial\left(\overline{\partial}X_a - \tfrac{1}{2}H_{abc}\,X^b\,\overline{\partial}X^c\right) + \overline{\partial}\left(\partial X_a - \tfrac{1}{2}H_{abc}\,\partial X^b\,X^c\right) \\ &= \partial\left(\overline{\partial}X_a - \tfrac{1}{2}H_{abc}\,X^b_{0,L}\,\overline{\partial}X^c_0\right) + \overline{\partial}\left(\partial X_a - \tfrac{1}{2}H_{abc}\,\partial X^b_0\,X^c_{0,R}\right) \\ &\quad - \tfrac{1}{2}H_{abc}\left[\partial\left(X^b_{0,R}\,\overline{\partial}X^c_0\right) + \overline{\partial}\left(\partial X^b_0\,X^c_{0,L}\right)\right] \\ &= \partial\left(\overline{\partial}X_a - \tfrac{1}{2}H_{abc}\,X^b_{0,L}\,\overline{\partial}X^c_0\right) + \overline{\partial}\left(\partial X_a - \tfrac{1}{2}H_{abc}\,\partial X^b_0\,X^c_{0,R}\right)\ ,\end{aligned} \tag{4.2.3}$$

where we exploited that we work linear order in the flux by rewriting the terms proportional to H in terms of the free fields and used the free field equations, i.e. $\partial X_{0,R} = \overline{\partial}X_{0,L} = 0$. By using the equations of motions (4.1.4), both terms in the above are also conserved separately and thus we have found the chiral currents

$$\boxed{\begin{aligned}J^a(z) &:= i\partial X^a(z,\overline{z}) - \frac{i}{2}H_{abc}\,\partial X^b_0(z)\,X^c_{0,R}(\overline{z}) \\ \overline{J}^a(\overline{z}) &:= i\overline{\partial}X^a(z,\overline{z}) - \frac{i}{2}H_{abc}\,X^b_{0,L}(z)\,\overline{\partial}X^c_0(\overline{z})\end{aligned}} \tag{4.2.4}$$

with $\overline{\partial}J^a = \partial\overline{J}^a = 0$. These currents could have also been found directly from the equations of motion as in the WZW model; cf. remark 1.

Remark 5. In the WZW model the the chiral currents correspond to the symmetry under conjugation of the fields g with elements of the group manifold. Here we are considering a three-torus $\mathbb{T}^3$ which is a compact abelian Lie group. We also mentioned the formal identification $\ln(g) \sim X$. Hence conjugating g with elements of an abelian group e^a looks like translating X by formally using the logarithm identities.

Remark 6. Upon inserting the classical solution (4.1.5) into (4.2.4) we observe that $J^a = J_0^a = i\partial X_0^a$ and $\overline{J}^a = J_0^a = i\overline{\partial} X_0^a$. That is, the chiral currents (4.2.4) in CFT$_H$ coincide with the currents of the free theory. However, this does not enable us to conclude that CFT$_H$ is merely the free bosonic CFT as this statement is only classically. Indeed, in the following we will see that the quantum theory CFT$_H$ differs considerably from the free bosonic CFT.

4.2.2 Conformal perturbation theory

Now we turn to the quantum theory. For calculating correlation functions we want to consider the H-term in the action (4.1.3) as a perturbation, i.e.

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1 \qquad \text{with} \qquad \mathcal{S}_1 = \frac{H_{abc}}{6\pi\alpha'} \int_\Sigma d^2z\, X^a\, \partial X^b\, \overline{\partial} X^c\,. \tag{4.2.5}$$

The n-point correlator of n operators $\mathfrak{O}_i(X)$ can be written as

$$\begin{aligned} \langle \mathfrak{O}_1 \dots \mathfrak{O}_n \rangle &= \frac{1}{\mathcal{Z}} \int [\mathcal{D}X]\, \mathfrak{O}_1 \dots \mathfrak{O}_n\, e^{-\mathcal{S}[X]} \\ &= \langle \mathfrak{O}_1 \dots \mathfrak{O}_n \rangle_0 - \langle \mathfrak{O}_1 \dots \mathfrak{O}_n \mathcal{S}_1 \rangle_0 - \langle \mathfrak{O}_1 \dots \mathfrak{O}_n \rangle_0 \langle \mathcal{S}_1 \rangle_0 + \mathcal{O}(H^2) \\ &= \langle \mathfrak{O}_1 \dots \mathfrak{O}_n \rangle_0 - \langle \mathfrak{O}_1 \dots \mathfrak{O}_n \mathcal{S}_1 \rangle_0 + \mathcal{O}(H^2)\,, \end{aligned} \tag{4.2.6}$$

where $\mathcal{Z} = \int [\mathcal{D}X] e^{-\mathcal{S}[X]}$ is the partition function and $\langle \dots \rangle_0$ denotes the correlator evaluated with respect to the free action $\mathcal{S}_0$. Moreover, $\langle \mathcal{S}_1 \rangle_0 = 0$ since the free action is second order in the fields which only allows for an *even number of operator insertions to be non-trivial*; $\mathcal{S}_1$ consists of an odd number of operators.

The free propagator with respect to which correlators will be computed is standard and reads [21]

$$\mathfrak{G}^{ab}(z,w) = \left\langle X^a(z,\overline{z})\, X^b(w,\overline{w}) \right\rangle = -\frac{\alpha'}{2} \ln|z-w|^2\, \delta^{ab}\,. \tag{4.2.7}$$

Remark 7. As an odd number of operator insertions vanishes, in particular the insertion of two fields X does not get any contributions linear order in H since $\langle X X \mathcal{S}_1 \rangle$ contains an odd number of fields. Thus the free propagator (4.2.7) is unperturbed up to $\mathcal{O}(H)$. However, in [20] second order corrections have been computed which reveals a dependence on a cut-off parameter, i.e. at this order we observe a renormalization group flow. This is reasonable since at $\mathcal{O}(H^2)$ the string beta-functionals (2.1.11) no longer vanish and the system has to flow to another conformal fix point. Indeed the correction was found to be proportional to $H^a{}_{cd} H^{cdb}$ as expected from (2.1.11).

4.2.3 Correlators of the currents

In this section we want to deduce the most important correlators of the currents (4.2.4). This comprises two- and three-point functions.

Two-point functions

Calculating the two-point functions is particularly easy since the term involving the perturbation $\mathcal{S}_1$ is odd as well as the terms involving H and therefore do not contribute. Using (4.2.7) we readily obtain

$$\begin{aligned}
\left\langle J^a(z_1)\, J^b(z_2)\right\rangle &= (i)^2\, \left\langle \partial_{z_1} X^a(z_1)\, \partial_{z_2} X^a(z_2)\right\rangle_0 = \frac{\alpha'}{2}\, \frac{\delta^{ab}}{(z_1 - z_2)^2} \\
\left\langle J^a(z_1)\, \overline{J}^b(z_2)\right\rangle &= (i)^2\, \left\langle \partial_{z_1} X^a(z_1)\, \overline{\partial}_{z_2} X^a(z_2)\right\rangle_0 = 0 \\
\left\langle \overline{J}^a(z_1)\, \overline{J}^b(z_2)\right\rangle &= (i)^2\, \left\langle \overline{\partial}_{z_1} X^a(z_1)\, \overline{\partial}_{z_2} X^a(z_2)\right\rangle_0 = \frac{\alpha'}{2}\, \frac{\delta^{ab}}{(\overline{z}_1 - \overline{z}_2)^2}\,.
\end{aligned} \tag{4.2.8}$$

Three-point functions

Computing three-point functions is a bit more involved and will be presented in more detail. Using (4.2.6) we have to evaluate

$$\left\langle J^a(z_1)\, J^b(z_2)\; J^c(z_3)\right\rangle = \left\langle J^a(z_1)\, J^b(z_2)\, J^c(z_3)\right\rangle_0 - \left\langle J^a(z_1)\, J^b(z_2)\, J^c(z_3)\, \mathcal{S}_1\right\rangle_0 \tag{4.2.9}$$

and similarly for the other combinations including $\overline{J}$. We will present the computation of the desired correlators in some detail. Our strategy is to compute correlators of appropriate combinations of the fields X first and then deduce all the possible current correlators by just differentiating. We will proceed successively and first consider

$$\begin{aligned}
K_0^{abc} := \; & \left\langle X^a(z_1, \overline{z}_1)\, X^b(z_2, \overline{z}_2)\, X^c(z_3, \overline{z}_3)\right\rangle_0 \\
& - \frac{1}{2} H^a{}_{pq} \left\langle : X^p_{0,L}(z_1)\, X^q_{0,R}(\overline{z}_1) : X^b(z_2, \overline{z}_2)\, X^c(z_3, \overline{z}_3)\right\rangle_0 \\
& - \frac{1}{2} H^b{}_{pq} \left\langle X^a(z_2, \overline{z}_2) : X^p_{0,L}(z_1)\, X^q_{0,R}(\overline{z}_1) : X^c(z_3, \overline{z}_3)\right\rangle_0 \\
& - \frac{1}{2} H^c{}_{pq} \left\langle X^a(z_2, \overline{z}_2)\, X^c(z_3, \overline{z}_3) : X^p_{0,L}(z_1)\, X^q_{0,R}(\overline{z}_1) :\right\rangle_0
\end{aligned} \tag{4.2.10}$$

with terms of higher order in H neglected. From this we will obtain the current correlators in the free theory. The first term vanishes since it is odd and the remaining three evaluate as follows. We have

$$\begin{aligned}
k^{pq;bc} := \; & \left\langle : X^p_{0,L}(z_1)\, X^q_{0,R}(\overline{z}_1) : X^b(z_2, \overline{z}_2)\, X^c(z_3, \overline{z}_3)\right\rangle_0 \\
& + \left\langle : X^p_{0,L}(z_1)\, X^q_{0,R}(\overline{z}_1) : X^b(z_2, \overline{z}_2)\, X^c(z_3, \overline{z}_3)\right\rangle_0 \\
= \; & \tfrac{(\alpha')^2}{4} \left[\ln(z_{12})\, \ln(\overline{z}_{13})\, \delta^{pb}\, \delta^{qc} + \ln(z_{13})\, \ln(\overline{z}_{12})\, \delta^{pc}\, \delta^{qb} \right] ,
\end{aligned} \tag{4.2.11}$$

where we denoted $z_{ij} := z_i - z_j$ and used (4.2.7) which can be split into a right-moving and a left-moving part by exploiting $\ln|z| = \ln(z) + \ln(\overline{z})$ for the principal branch of the logarithm. Moreover, since this term is already linear in H, all fields can be considered free. Proceeding analogously for the other two terms we conclude

$$\begin{aligned} K_0^{abc} &= -\frac{(\alpha')^2}{8}\left(H^a{}_{pq}\,k^{pq;bc} + H^b{}_{pq}\,k^{a;pq;c} + H^c{}_{pq}\,k^{ab;pq}\right) \\ &= -\frac{(\alpha')^2}{8} H^{abc}\big[\ln(z_{12})\,\ln(\overline{z}_{13}) - \ln(z_{13})\,\ln(\overline{z}_{12}) - \ln(z_{12})\,\ln(\overline{z}_{23}) \\ &\qquad\qquad + \ln(z_{23})\,\ln(\overline{z}_{12}) + \ln(z_{13})\,\ln(\overline{z}_{23}) - \ln(z_{23})\,\ln(\overline{z}_{13})\big]\,. \end{aligned} \tag{4.2.12}$$

From K and (4.2.4) we readily obtain

$$\begin{aligned} \big\langle J^a(z_1)\,J^b(z_2)\,J^c(z_3)\big\rangle_0 &= -i\,\partial_{z_1}\partial_{z_2}\partial_{z_3}K_0^{abc} = 0 \\ \big\langle J^a(z_1)\,J^b(z_2)\,\overline{J}^c(\overline{z}_3)\big\rangle_0 &= -i\,\partial_{z_1}\partial_{z_2}\overline{\partial}_{z_3}K_0^{abc} = +\frac{i(\alpha')^2}{8}H^{abc}\,\frac{\overline{z}_{12}}{z_{12}^2\,\overline{z}_{13}\,\overline{z}_{23}} \\ \big\langle \overline{J}^a(\overline{z}_1)\,\overline{J}^b(\overline{z}_2)\,J^c(z_3)\big\rangle_0 &= -i\,\overline{\partial}_{z_1}\overline{\partial}_{z_2}\partial_{z_3}K_0^{abc} = -\frac{i(\alpha')^2}{8}H^{abc}\,\frac{z_{12}}{\overline{z}_{12}^2\,z_{13}\,z_{23}} \\ \big\langle J^a(z_1)\,J^b(z_2)\,J^c(z_3)\big\rangle_0 &= -i\,\partial_{z_1}\partial_{z_2}\partial_{z_3}K_0^{abc} = 0\,. \end{aligned} \tag{4.2.13}$$

To obtain the perturbed contribution to the full $\mathcal{O}(H)$ current correlator we compute

$$\begin{aligned} K_1^{abc} :&= \frac{H_{pqr}}{6\pi\alpha'}\int_\sigma d^2w\big\langle X^a(z_1,\overline{z}_1)\,X^b(z_2,\overline{z}_2)\,X^c(z_3,\overline{z}_3) \\ &\qquad\qquad \times :X^p(w,\overline{w})\,\partial X^q(w)\,\overline{\partial}X^r(\overline{w}):\big\rangle_0 \\ &= -\frac{(\alpha')^2}{48\pi}\,H^{abc}\int_\Sigma d^2w\Bigg[\left(\frac{\ln|z_1-w|^2}{(z_2-w)(\overline{z}_3-\overline{w})} - \frac{\ln|z_1-w|^2}{(z_3-w)(\overline{z}_2-\overline{w})}\right) \\ &\qquad + \left(\frac{\ln|z_2-w|^2}{(z_3-w)(\overline{z}_1-\overline{w})} - \frac{\ln|z_2-w|^2}{(z_1-w)(\overline{z}_3-\overline{w})}\right) \\ &\qquad + \left(\frac{\ln|z_3-w|^2}{(z_1-w)(\overline{z}_2-\overline{w})} - \frac{\ln|z_3-w|^2}{(z_2-w)(\overline{z}_1-\overline{w})}\right)\Bigg]\,, \end{aligned} \tag{4.2.14}$$

where we performed all six contractions using (4.2.7) and the antisymmetry of H as above. As we did it above, all the correlators including the perturbation $\mathcal{S}_1$ can be obtained by differentiating K_1 and additionally by using the identity

$$\partial_z\,\frac{1}{\overline{z}} = \overline{\partial}_z\,\frac{1}{z} = 2\pi\,\delta^{(2)}(z) \tag{4.2.15}$$

which has to be understood in the sense of distributions. The strategy is to arrange the integrand of (4.2.14) if necessary via integration by parts such that a δ-distribution of the form $\delta^{(2)}(z_i - w)$ appears once in each term. We will illustrate this, say, with the first term.

$$\partial_{z_1}\partial_{z_2}\overline{\partial}_{z_3}\int_\Sigma d^2w\,\frac{\ln|z_1-w|^2}{(z_2-w)(\overline{z}_3-\overline{w})}\,. \tag{4.2.16}$$

This combination of derivatives does not generate a δ. But the anti-holomorphic factor in the denominator can be rewritten as $(\overline{z}_3 - \overline{w})^{-1} = -\overline{\partial}_w \ln|z_3 - w|^2$ and we can perform a partial integration to have the $\overline{w}$-derivative acting on the other logarithm in the numerator. Thus

$$\begin{aligned} \partial_{z_1}\partial_{z_2}\overline{\partial}_{z_3}\int_\Sigma d^2w\,\frac{\ln|z_1-w|^2}{(z_2-w)(\overline{z}_3-\overline{w})} &= -\partial_{z_1}\partial_{z_2}\overline{\partial}_{z_3}\int_\Sigma d^2w\,\frac{\ln|z_3-w|^2}{(z_2-w)(\overline{z}_1-\overline{w})} \\ &= \int_\Sigma d^2w\,\frac{2\pi\,\delta^{(2)}(z_1-w)}{(z_3-w)(z_2-w)^2} \\ &= \frac{2\pi}{z_{31}z_{21}^2}\,. \end{aligned} \tag{4.2.17}$$

This allows us to compute

$$\begin{aligned} \left\langle J^a(z_1)\,J^b(z_2)\,J^c(z_3)\,\mathcal{S}_1\right\rangle_0 &= -i\,\partial_{z_1}\partial_{z_2}\partial_{z_3}K_1^{abc} = +\frac{i(\alpha')^2}{8}\,\frac{H^{abc}}{z_{12}\,z_{13}\,z_{23}} \\ \left\langle J^a(z_1)\,J^b(z_2)\,\overline{J}^c(z_3)\,\mathcal{S}_1\right\rangle_0 &= -i\,\partial_{z_1}\partial_{z_2}\overline{\partial}_{z_3}K_1^{abc} = +\frac{i(\alpha')^2}{8}\,H^{abc}\,\frac{\overline{z}_{12}}{z_{12}^2\,\overline{z}_{13}\,\overline{z}_{23}} \\ \left\langle \overline{J}^a(z_1)\,\overline{J}^b(z_2)\,J^c(z_3)\,\mathcal{S}_1\right\rangle_0 &= -i\,\overline{\partial}_{z_1}\overline{\partial}_{z_2}\partial_{z_3}K_1^{abc} = -\frac{i(\alpha')^2}{8}\,H^{abc}\,\frac{z_{12}}{\overline{z}_{12}^2\,z_{13}\,z_{23}} \\ \left\langle \overline{J}^a(z_1)\,\overline{J}^b(z_2)\,\overline{J}^c(z_3)\,\mathcal{S}_1\right\rangle_0 &= -i\,\overline{\partial}_{z_1}\overline{\partial}_{z_2}\overline{\partial}_{z_3}K_1^{abc} = -\frac{i(\alpha')^2}{8}\,\frac{H^{abc}}{\overline{z}_{12}\,\overline{z}_{13}\,\overline{z}_{23}}\,. \end{aligned} \tag{4.2.18}$$

The final result is now obtained by subtracting (4.2.13) and (4.2.18) which yields

$$\boxed{\begin{aligned} \left\langle J^a(z_1)\,J^b(z_2)\,J^c(z_3)\right\rangle &= -\frac{i(\alpha')^2}{8}\,\frac{H^{abc}}{z_{12}\,z_{13}\,z_{23}} \\ \left\langle J^a(z_1)\,J^b(z_2)\,\overline{J}^c(z_3)\right\rangle &= 0 \\ \left\langle \overline{J}^a(z_1)\,\overline{J}^b(z_2)\,J^c(z_3)\right\rangle &= 0 \\ \left\langle \overline{J}^a(z_1)\,\overline{J}^b(z_2)\,\overline{J}^c(z_3)\right\rangle &= +\frac{i(\alpha')^2}{8}\,\frac{H^{abc}}{\overline{z}_{12}\,\overline{z}_{13}\,\overline{z}_{23}} \end{aligned}} \tag{4.2.19}$$

In particular we see that the currents (4.2.4) have holomorphic respectively anti-holomorphic correlation functions.

4.2.4 The current algebra and the energy-momentum tensor

We want to find the OPE of the chiral currents and the energy momentum tensor of the theory. The OPE of the currents could in principle be directly read-off from the general current algebra of a WZW model (2.2.20) since we know that the structure constant is proportional to H. However, so far we developed the CFT very closely to the discussion of the WZW model. Therefore we expect the currents (4.2.4) to be primary fields of conformal weight $h = 1$ or $\overline{h} = 1$ in our CFT respectively. Thus the current algebra can also be determined from the structure of the two- and three-point functions. In general, the JJ-OPE is of the form [35]

$$J^a(z)\, J^b(w) = \sum_{k,n\geq 0} C_{JJ}{}^{\Phi_k}\, \frac{a^n_{JJ\Phi_k}}{n!}\, \frac{\partial^n_w \Phi_k(w)}{(z-w)^{2-h_k-n}} \tag{4.2.20}$$

provided J has conformal weight $h = 1$. The coefficients $C_{JJ}{}^k$ can be determined from the two- and three-point functions since we know that for general chiral primaries Φ_i of weight h_i

$$\begin{aligned} \langle \Phi_1(z)\, \Phi_2(w) \rangle &= \frac{d_{\Phi_1\Phi_2}\, \delta_{h_1,h_2}}{(z-w)^{h_1+h_2}} \\ \langle \Phi_1(z_1)\, \Phi_2(z_2)\, \Phi_3(z_3) \rangle &= \frac{C_{\Phi_1\Phi_2\Phi_3}}{z_{12}^{h_1+h_2-h_3}\, z_{13}^{h_1+h_3-h_2}\, z_{23}^{h_2+h_3-h_1}} \end{aligned} \tag{4.2.21}$$

and the OPE coefficients are related to the three-point coefficients by $C_{\Phi_i\Phi_j\Phi_k} = C_{\Phi_i\Phi_j}{}^{\Phi_l}$ $(d^{-1})_{\Phi_l\Phi_k}$. J will be the only primary of weight $h = 1$ in our theory. Therefore we are just interested in $n = 0$ such that $a^0_{JJ\Phi_k} = 1$ and only have to consider the J correlators as well as $\mathbb{1}$ which has weight 0 and $C_{JJ}{}^{\mathbb{1}} = d_{JJ}$.

From (4.2.8) and (4.2.19) we read-off $d_{JJ} = \alpha'/2\, \delta^{ab}$ and $C_{JJJ} = -i(\alpha')^2/8\, H^{abc}$. Thus the only non-vanishing OPE coefficients read $C_{JJ}{}^{\mathbb{1}} = \alpha'/2\, \delta^{ab}$ $C_{JJ}{}^J = -i\alpha'/4\, H^{abc}$. Doing the same for the anti-holomorphic currents and using (4.2.20) we find the singular parts of the OPE's

$$\begin{aligned} J^a(z)\, J^b(w) &= \frac{\alpha'}{2}\, \frac{\delta^{ab}}{(z-w)^2} - i\frac{\alpha'}{4}\, \frac{H^{ab}{}_c}{z-w}\, J^c(w) + \text{reg.} \\ \overline{J}^a(\overline{z})\, \overline{J}^b(\overline{w}) &= \frac{\alpha'}{2}\, \frac{\delta^{ab}}{(\overline{z}-\overline{w})^2} + i\frac{\alpha'}{4}\, \frac{H^{ab}{}_c}{\overline{z}-\overline{w}}\, \overline{J}^c(\overline{w}) + \text{reg.} \end{aligned} \tag{4.2.22}$$

and the $J\overline{J}$-OPE is purely regular. This is indeed the current algebra of a WZW model with $f^{abc} = -\alpha'/4\, H^{abc}$ and $k = \alpha'/2$[2].

As we have showed in subsection 2.2.3, from the current algebra (4.2.22) one can also deduce the Kač-Moody algebra (2.2.23). Laurent-expanding the currents and employing the same techniques we find for the modes

$$\begin{aligned} [\, j^a_m, j^b_n\,] &= -i\tfrac{\alpha'}{4} H^{ab}{}_c\, j^c_{m+n} + m\, \delta^{ab}\, \delta_{m,-n} \\ [\, \overline{j}^a_m, \overline{j}^b_n\,] &= +i\tfrac{\alpha'}{4} H^{ab}{}_c\, \overline{j}^c_{m+n} + m\, \delta^{ab}\, \delta_{m,-n}\,. \end{aligned} \tag{4.2.23}$$

Remark 8. In a WZW model the OPE of the chiral currents does not have the relative sign in the second term in (4.2.22) as well as in the Kač-Moody algebra (4.2.23). This is simply a matter of definition; by defining $J \to -J$, from (4.2.19) we observe that the relative sign would be absent.

[2]If we would have introduced α' in the WZW model (2.2.2) we would have found a quantization condition for k in terms of α'.

The energy-momentum tensor

Following the lines of the discussion of the WZW model in section 2.2, the energy-momentum tensor (2.2.24) was determined by the Sugawara construction; in the CFT$_H$ the quantum energy-momentum tensor thus suitably normalized reads

$$T(z) = \frac{\delta_{ab}}{\alpha'} : J^a \, J^b : (z) \quad , \quad \overline{T}(z) = \frac{\delta_{ab}}{\alpha'} : \overline{J}^a \, \overline{J}^b : (z) \, . \tag{4.2.24}$$

The normalization can be justified by calculating the TT-OPE.

How to contract. We already defined a generalized Wick theorem (2.2.26) in order to contract correctly in interacting theories. However, as we already observed in remark 3, contractions up to linear order in the structure constants, H in our case, are captured correctly by just replacing the contracted fields by the respective OPE deduced from the current algebra, here (4.2.22), without writing-out normal orderings.

Using the OPE (4.2.22) and employing the antisymmetry of H yields the TT-OPE

$$\begin{aligned} T(z)\, T(w) &= \frac{c/2}{(z-w)^4} + \frac{2\, T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{reg.} \\ \overline{T}(\overline{z})\, \overline{T}(\overline{w}) &= \frac{c/2}{(\overline{z}-\overline{w})^4} + \frac{2\, \overline{T}(\overline{w})}{(\overline{z}-\overline{w})^2} + \frac{\overline{\partial} \overline{T}(\overline{w})}{(\overline{z}-\overline{w})} + \text{reg.} \end{aligned} \tag{4.2.25}$$

with the central charge $c = 3$ which is the same as in the free theory since we only consider the three directions of the torus.

Since we deduced the energy-momentum tensor in CFT$_H$ we are able to identify the primary fields in the theory. In particular, the currents (4.2.4) should be chiral primaries of conformal weight $(h, \overline{h}) = (1, 0)$ respectively $(h, \overline{h}) = (0, 1)$. Indeed, by again employing (4.2.22)and the antisymmetry of H we readily find

$$\begin{aligned} T(z)\, J^a(w) &= \frac{J^a(w)}{(z-w)^2} + \frac{\partial_w\, J^a(w)}{(z-w)} + \text{reg.} \\ \overline{T}(\overline{z})\, J^a(w) &= \text{reg.} \\ T(z)\, \overline{J}^a(\overline{w}) &= \text{reg.} \\ \overline{T}(\overline{z})\, \overline{J}^a(\overline{w}) &= \frac{\overline{J}^a(\overline{w})}{(\overline{z}-\overline{w})^2} + \frac{\overline{\partial}_w\, \overline{J}^a(\overline{w})}{(\overline{z}-\overline{w})} \text{reg.}\, , \end{aligned} \tag{4.2.26}$$

i.e. J and $\overline{J}$ are chiral primaries with the respective conformal weights.

4.2.5 Vertex operators

In string theory the states one intends to scatter are represented by *vertex operators* since the bosonic fields X itself are not conformal. In order to respect all the symmetries of our theory, the vertex operators have to have conformal weight $(h, \overline{h}) = (1, 1)$. As we would like do deduce properties of the spacetime geometry from CFT correlators analogously to the open string case, we of course have to identify the proper objects to scatter. Since we are dealing with an interacting theory, we cannot expect the vertex operators of the free theory still to be valid. In the following we will investigate the tachyon vertex operator.

Proper "coordinates" in CFT_H

From a conformal field theory perspective the currents (4.2.4) are the proper fields to study since they have a well-defined behavior under conformal transformations. In a free theory the bosonic coordinates $X_{L/R}$ are basically the primitives of the currents and this is why one is able to make statements about the bosonic fields with CFT methods. However, this does not apply to our theory as one can see from (4.2.4). Thus the proper "coordinates", denoted $\mathcal{X}(z,\overline{z})$, to consider in this setup are defined as integrals of (4.2.4), i.e. by

$$J^a(z) =: i\,\partial\mathcal{X}^a(z) \quad , \quad \overline{J}^a(\overline{z}) =: i\,\overline{\partial}\mathcal{X}^a(\overline{z})\,. \tag{4.2.27}$$

In particular, due to the anti-/holomorphicity of the currents these fields obey the wave equation $\overline{\partial}\partial\mathcal{X}^a = 0$ and thus split into a left- and right-moving part

$$\mathcal{X}^a(z,\overline{z}) = \mathcal{X}^a_L(z) + \mathcal{X}^a_R(\overline{z}) \tag{4.2.28}$$

as this is the most general solution.

Remark 9. Let us collect some evidence for an interpretation as spacetime coordinates of this fields. As can be seen from remark 6, linear order in H we have $\mathcal{X}^a_{L/R} = X^a_{0,L/R}$ since the free fields are the primitives of the currents. These are of course coordinates for the spacetime manifold since they are in the free theory $H = 0$. It is crucial at this point to interpret this result correctly: Also for the free theory, the manifold is given by $\mathbb{T}^3$ and all we want to know about the $\mathcal{X}$'s is whether they are still coordinates for the torus – they are! However, the complete geometry is also determined by the metric and the torsion and as we have already seen in current algebra, they also have a strong impact on the CFT. Thus the spacetime geometry is nevertheless $(\mathbb{T}^3, G, H)$ at $\mathcal{O}(H)$; with $\mathcal{X}$ we have just chosen a different atlas.

For later reference we also need the OPE of $\mathcal{X}$ with the currents (4.2.4). These can be obtained by formally integrating the JJ-OPE (4.2.22), giving

$$\begin{aligned} J^a(z)\mathcal{X}^b_L(w) &= -i\frac{\alpha'}{2}\,\frac{\delta^{ab}}{(z-w)} + \frac{\alpha'}{4}\,H^{ab}{}_c\,J^c(w)\,\ln(z-w) + \text{reg.} \\ \overline{J}^a(\overline{z})\mathcal{X}^b_R(\overline{w}) &= -i\frac{\alpha'}{2}\,\frac{\delta^{ab}}{(\overline{z}-\overline{w})} - \frac{\alpha'}{4}\,H^{ab}{}_c\,\overline{J}^c(\overline{w})\,\ln(\overline{z}-\overline{w}) + \text{reg.} \end{aligned} \tag{4.2.29}$$

and the other combinations vanish. We (formally) used integration by parts to obtain the logarithmic term together with regular ones. Furthermore we neglected the possibility of integration constants here which can be either holomorphic or anti-holomorphic functions; this freedom becomes important below.

The tachyon vertex operator

After identifying all the necessary ingredients, we can now define the *tachyon vertex operator* analogous to the free theory as

$$V(z,\overline{z}) := :\exp\left(i\,k_L\cdot\mathcal{X}_L + i\,k_R\cdot\mathcal{X}_R\right): \tag{4.2.30}$$

with the momenta (4.1.9) and the shorthand notation $k\cdot\mathcal{X} := \delta_{ab}k^a\mathcal{X}^b$. Of course it has to be justified to be called vertex operator. By employing the JX-OPE (4.2.29) we first compute

$$\begin{aligned}
J^a(z)V(w,\overline{w}) &= J^a(z)\,:\exp\left(i\,k_L\cdot\mathcal{X}_L + i\,k_R\cdot\mathcal{X}_R\right):(w,\overline{w})\\
&= \sum_{n=0}^{\infty}\frac{(ik_{L\,b})^n}{n!}\,J^a(z)\,:(\mathcal{X}_L^b)^n\,e^{ik_R\cdot\mathcal{X}_R}:\\
&= ik_{L\,b}\sum_{n=0}^{\infty}\frac{(ik_{L\,c})^n}{n!}\,n\,:\overbrace{J^a(z)\,\mathcal{X}_L^b}\,(\mathcal{X}^c)^n\,e^{ik_R\cdot\mathcal{X}_R}:\\
&= \frac{\alpha'\,k_L^a}{2}\,\frac{V(w,\overline{w})}{(z-w)} + i\frac{\alpha'}{4}\,H^a{}_{bc}\,k_L^b\ :J^c\,V:(w,\overline{w})\ \ln(z-w) + \text{reg.}
\end{aligned} \tag{4.2.31}$$

and similarly

$$\overline{J}^a(\overline{z})V(w,\overline{w}) = \frac{\alpha'\,k_R^a}{2}\,\frac{V(w,\overline{w})}{(\overline{z}-\overline{w})} - i\frac{\alpha'}{4}\,H^a{}_{bc}\,k_R^b\ :\overline{J}^c\,V:(w,\overline{w})\ \ln(\overline{z}-\overline{w}) + \text{reg.}\,. \tag{4.2.32}$$

These can be used to compute the OPE with the energy momentum tensor (4.2.24). We obtain

$$\begin{aligned}
T(z)V(w,\overline{w}) &= \frac{\delta_{ab}}{2}\ :J^aJ^b:(z)\,V(w,\overline{w})\\
&= 2\,\frac{\delta_{ab}}{\alpha'}:\Bigg[\left(\frac{\alpha'}{2}\,\frac{k_L^a}{(z-w)} + i\,\frac{\alpha'}{4}\,H^a{}_{pq}\,k_L^p\,J^q(z)\,\ln(z-w)\right)J^b(z)\\
&\quad + \left(\frac{(\alpha')^2}{4}\,\frac{k_L^a\,k_L^b}{(z-w)^2} + i\,\frac{\alpha'}{2}\,H^a{}_{pq}\,k_L^p\,k_L^b\,J^q(z)\,\frac{\ln(z-w)}{(z-w)}\right)\Bigg]\\
&\quad \times V(w,\overline{w}): + \text{reg.}\\
&= \frac{\alpha'\,k_L^2}{4}\,\frac{V(w,\overline{w})}{(z-w)^2} + \frac{k_L^a\,:J_a\,V:(w,\overline{w})}{(z-w)} + \text{reg.}\\
&= \frac{\alpha'\,k_L^2}{4}\,\frac{V(w,\overline{w})}{(z-w)^2} + \frac{\partial_w V(w,\overline{w})}{(z-w)} + \text{reg.}\,,
\end{aligned} \tag{4.2.33}$$

where the antisymmetry of H was exploited. Similarly we find

$$\overline{T}(\overline{z})V(w,\overline{w}) = \frac{\alpha'\,k_R^2}{4}\,\frac{V(w,\overline{w})}{(\overline{z}-\overline{w})^2} + \frac{\overline{\partial}_w V(w,\overline{w})}{(\overline{z}-\overline{w})} + \text{reg.}\,. \tag{4.2.34}$$

Thus V is a primary field of weight $(\frac{\alpha'}{4}k_L^2, \frac{\alpha'}{4}k_R^2)$ which has to be equal to $(1,1)$ in order to define a vertex operator. Therefore we demand $\alpha' k_{L/R}^2 = 4$ which is the usual mass-shell condition, i.e. V indeed corresponds to the tachyon.

The vertex operators – local operators in a CFT – correspond one-to-one to states in the theory. In particular, the tachyon vertex operator in the free theory corresponds to the the degenerate ground state $|k_L, k_R\rangle$. In the following we want to discuss the properties of the operator corresponding to V in CFT$_H$, that is, the state

$$|V\rangle := \int [\mathcal{D}X]\; V(z=0,\overline{z}=0)\, e^{-S[X]} = \lim_{z,\overline{z}\to 0} V(z,\overline{z})|0\rangle \tag{4.2.35}$$

where $|0\rangle$ denotes the ground state of the theory corresponding to the identity operator. If we would Laurent-expand the fields $\partial\mathcal{X}(z)$ and $\overline{\partial}\mathcal{X}(\overline{z})$ and declaring the appearing coefficients to be creation and annihilation operators, it is a standard argument to show that (4.2.35) corresponds to a vacuum[3]. Rather we want to investigate if $|V\rangle$ carries momentum respectively winding as in the free theory. Therefore we have to act with the momentum operator (4.1.11) on the state; this can be done as follows

$$\begin{aligned} P_L^a\,|V\rangle &= \lim_{w,\overline{w}\to 0} P_L^a\, V(w,\overline{w})|0\rangle \\ &= \lim_{w,\overline{w}\to 0} \frac{1}{\alpha'} \oint_{C(0)} \frac{dz}{2\pi i}\, i\partial X^a(z,\overline{z})\, V(w,\overline{w})|0\rangle \\ &= \lim_{w,\overline{w}\to 0} \frac{1}{\alpha'} \oint_{C(0)} \frac{dz}{2\pi i}\; :\left[J^a(z) + \frac{i}{2}\, H^a{}_{bc}\, \partial X^b_{0,L}(z)\, X^c_{0,R}(\overline{z})\right]:\; V(w,\overline{w})|0\rangle\,. \end{aligned} \tag{4.2.36}$$

In the last step we observed that ∂X^a is not a proper field in CFT$_H$ but that the OPE can be computed by using the OPE with the holomorphic current and subtracting the difference $J^a - i\partial X^a$ according to the definition (4.2.4). The subtracted term can be seen to be linear in H already and thus the OPE of this term with V reduces to the OPE with the free tachyon vertex operator $V_0(w,\overline{w}) = \exp(ik_L\cdot X_{0,L} + ik_R\cdot X_{0,R})$ in the free CFT. That is, the second term can be evaluated just by substituting the free propagator (4.2.7) for any contraction as usual. The evaluation is straight forward in particular since we already encountered all necessary techniques. Together with (4.2.31) we obtain

$$\begin{aligned} P_L^a\,|V\rangle = \frac{k_L^a}{2} &\lim_{w,\overline{w}\to 0} V(w,\overline{w})|0\rangle \\ &\lim_{w,\overline{w}\to 0} \oint_{C(0)} \frac{dz}{2\pi i}\, \frac{\alpha'}{2}\, H^a{}_{bc} \Bigg[\frac{ik_L^b}{2}\; :J^c\, V:(w,\overline{w})\, \ln(z-w) \\ &+ \frac{k_L^b}{2}\, \frac{:X^c_{0,R}(\overline{z}) V_0(w,\overline{w}):}{(z-w)} + \frac{k_R^c}{2}\; :J_0^b\, V_0(w,\overline{w}):\ln(\overline{z}-\overline{w}) \\ &- \frac{i\alpha' k_L^b k_R^c}{4}\, \frac{V_0(w,\overline{w})\ln(\overline{z}-\overline{w})}{(z-w)} + \text{reg.}\Bigg]|0\rangle \end{aligned} \tag{4.2.37}$$

[3] See for example [36]

The first term gives the momentum eigenvalue. For scattering amplitudes and the momentum conservation appearing there to make sense, we require $|V\rangle$ to be an eigenstate for the momentum-operator. Thus all the remaining terms have to vanish. This can be achieved as follows.

- We can add a term $\sim H^a{}_{bc}\, k_L^b :J_0^c\, V_0:(w,\overline{w})\ \ln(\overline{z}-\overline{w})$ as integration constant to (4.2.29) since it is regular in z. This adds to the first term in the second line to give $\ln|z-w|^2$; this disappears after integration.
- Inserting the mode expansion (4.1.8), the only term in the expansion of $X^c_{0,R}$ which gives a non-vanishing contribution after integration is the zero-mode x^c since V_0 is non-singular. This gives a Hx-contribution that can be gauged away.
- The second logarithmic term can be treated analogously to the first.
- We encountered a term proportional to $H^a{}_{bc}k_L^b k_R^c = -\frac{2}{\alpha'}H^a{}_{bc}p^b w^c$ which survives even after performing the integral and the limit. It vanishes upon demanding

$$H^a{}_{bc}\, p^b\, w^c \simeq [\,\vec{p}\times\vec{w}\,] = 0\,. \tag{4.2.38}$$

  This condition was already encountered in subsection 4.1.1 by requiring the classical field to have winding w.

We have seen that all the remaining terms vanish upon using (4.2.38). Therefore we showed

$$P_L^a\,|V\rangle = \frac{k_L^a}{2}\,|V\rangle \tag{4.2.39}$$

and can argue similarly for the right-moving part. Hence we verified that $|V\rangle = |k_L, k_R\rangle$.

Remark 10. One may wonder why we haven't just imposed the constraint $H^a{}_{bc}k^b_{L/R} = 0$ since this would take care of the logarithmic terms we treated by adding integration constants to the JX-OPE. However, momenta transversal to the flux would trivialize most of our results. Since we also want to consider T-dualities later, it was argued in [7] that this conditions should be discarded in order to have sensible transformations. Nevertheless, the appearance of logarithmic terms in the OPE's of our theory may hint towards a *logarithmic CFT*[4].

4.3 T-dual backgrounds

In our aim of detecting interesting (non-)geometries in closed string theory we will not only allow for the H-flux background but also for the T-dual backgrounds identified in subsection 2.3.2. Above we have developed the CFT describing the sigma model (4.1.3) linear order in the flux which enables us to apply T-duality as discussed in section 2.3.

[4]See for example [37] and references therein.

Since we identified $\mathcal{X}^a$ in CFT$_H$ as proper coordinates in our theory which also split into a holomorphic and an anti-holomorphic parts, we recall the action of T-duality applicable in our case:

- The right-moving coordinates will be reflected, i.e.

$$\mathcal{X}^a(z,\bar{z}) = \mathcal{X}^a_L(z) + \mathcal{X}^a_R(\bar{z}) \quad \xrightarrow{T} \quad \widetilde{\mathcal{X}}^a(z,\bar{z}) = \mathcal{X}^a_L(z) - \mathcal{X}^a_R(\bar{z}) \,. \tag{4.3.1}$$

  In particular, T-duality acts by reflecting the anti-holomorphic current $\overline{J}^a$. Thus *an odd number of T-dualities changes the sign of the anti-holomorphic three-point function in* (4.2.19).

- The reflection exchanges momentum and winding modes, i.e.

$$p^a \quad \overset{T}{\longleftrightarrow} \quad w^a \,. \tag{4.3.2}$$

In the previous section we deduced the condition (4.2.38) $\vec{p} \times \vec{w} = \vec{0}$ in order for tachyon vertex operator V to have the usual momentum and winding quantum numbers. This condition is only satisfied if we just consider pure momentum or pure winding quantum numbers for V in the H-flux background. Recalling the definition (4.2.30) of V, pure winding scattering amounts to a relative sign in the exponent. The possible configurations in view of T-duality are given in Table 4.1. Since (4.2.38) is a condition for the

	H-flux		ω-flux		Q-flux		R-flux
i.	(p_1,p_2,p_3)	$\overset{T_3}{\longleftrightarrow}$	(p_1,p_2,w_3)	$\overset{T_2}{\longleftrightarrow}$	(p_1,w_2,w_3)	$\overset{T_1}{\longleftrightarrow}$	(w_1,w_2,w_3)
ii.	(p_1,p_2,w_3)	$\overset{T_3}{\longleftrightarrow}$	(p_1,p_2,p_3)	$\overset{T_2}{\longleftrightarrow}$	(p_1,w_2,p_3)	$\overset{T_1}{\longleftrightarrow}$	(w_1,w_2,p_3)
iii.	(p_1,w_2,w_3)	$\overset{T_3}{\longleftrightarrow}$	(p_1,w_2,p_3)	$\overset{T_2}{\longleftrightarrow}$	(p_1,p_2,p_3)	$\overset{T_1}{\longleftrightarrow}$	(w_1,p_2,p_3)
iv.	(w_1,w_2,w_3)	$\overset{T_3}{\longleftrightarrow}$	(w_1,w_2,p_3)	$\overset{T_2}{\longleftrightarrow}$	(w_1,p_2,p_3)	$\overset{T_1}{\longleftrightarrow}$	(p_1,p_2,p_3)

Table 4.1: The possible momentum and winding configurations for V with allowed and forbidden configurations due to (4.2.38) highlighted.

H-flux background, the admissible configurations for the T-dual backgrounds are the ones related to the admissible configurations with H-flux.

The pure momentum configurations are the most interesting ones as low-energy effective actions would be derived with respect to these. Hence we will focus on the admissible pure momentum configurations.

We only consider pure momentum scattering in the H- and R-flux background. The latter is defined by pure winding scattering in the H-flux background. The unknown spacetime for the R-flux will be denoted $\mathbb{T}R^3$ with coordinates $(\widetilde{\mathcal{X}}^a)$.

4.4 Tachyon scattering amplitudes

As we have already seen in chapter 3, CFT correlation functions can be used to study properties of the geometry of the theory. In the following we will follow the same spirit by investigating the structure of pure momentum N-point correlation functions of the previously discussed tachyons in the H- and R-flux background.

4.4.1 The basic three-point function and propagators

As already discussed in remark (7) the propagator (4.2.7) for the bosonic fields X^a is not corrected in the present interacting theory up to linear order in H. The perturbation at $\mathcal{O}(H)$ first affects the three-point function to be discussed and translated to the fields $\mathcal{X}^a$ here.

As discussed above, the basic fields in our theory are $\mathcal{X}^a$. The propagator with respect to these can be derived by using that $\mathcal{X}^a$ coincide with the free fields and the propagator (4.2.7). Upon using (4.1.5) we find

$$\begin{aligned}\big\langle \mathcal{X}^a(z,\overline{z})\,\mathcal{X}^b(w,\overline{w})\big\rangle &= \big\langle X_0^a(z,\overline{z})\,X_0^b(w,\overline{w})\big\rangle \\ &= \big\langle [X^a(z,\overline{z}) - \tfrac{1}{2}\,H^a{}_{pq}\,X^p_{0,L}(z)\,X^q_{0,R}(\overline{z})] \\ &\qquad \times [X^b(w,\overline{w}) - \tfrac{1}{2}\,H^b{}_{pq}\,X^p_{0,L}(w)\,X^q_{0,R}(\overline{w})]\big\rangle \\ &= \mathfrak{G}^{ab}(z,w) - \tfrac{1}{2}\big\langle X^a(z,\overline{z})\,H^b{}_{pq}\,X^p_{0,L}(w)\,X^q_{0,R}(\overline{w})\big\rangle_0 \\ &\qquad\qquad - \tfrac{1}{2}\big\langle H^b{}_{pq}\,X^p_{0,L}(z)\,X^q_{0,R}(\overline{z})\,X^a(w,\overline{w})\big\rangle_0 \\ &= \mathfrak{G}^{ab}(z,w)\end{aligned} \tag{4.4.1}$$

since we work up to linear order in the flux and three-point functions in the free theory trivially vanish. Thus the propagator is the same as (4.2.7);

$$\big\langle \mathcal{X}^a(z,\overline{z})\,\mathcal{X}^b(w,\overline{w})\big\rangle = -\frac{\alpha'}{2}\ln|z-w|^2\,\delta^{ab}\,. \tag{4.4.2}$$

The perturbation can be captured by considering the three-point function of the basic fields $\mathcal{X}$. By definition of $\mathcal{X}^a$ this can be determined by integrating (4.2.19). In order to proceed, we need to introduce the Rogers dilogarithm [38, 39, 40].

Definition 5. The *Rogers dilogarithm* is defined by

$$L(z) := -\frac{1}{2}\int_0^1 dx\,\left[\frac{\ln(1-z\,x)}{x} + \frac{\ln(z\,x)}{z^{-1}-x}\right] \tag{4.4.3}$$

for all $z \in \mathbb{C}\setminus\{0,1\}$.

With this definition at hand we can verify the following result.

Lemma 6. We have

$$\frac{1}{z_{12}\,z_{13}\,z_{23}} = \frac{2}{3}\,\partial_{z_1}\partial_{z_2}\partial_{z_3}\left[L\left(\frac{z_{12}}{z_{13}}\right) + L\left(\frac{z_{23}}{z_{21}}\right) + L\left(\frac{z_{13}}{z_{23}}\right)\right] \tag{4.4.4}$$

with $z_{ij} := z_i - z_j$ and analogously for $\overline{z}_{ij}$.

Proof. First we will compute the derivative of (4.4.3). We find

$$\frac{dL(z)}{dz} = -\frac{1}{2}\int_0^1 dx\, \frac{\ln(z\,x)}{(1-z\,x)^2} = -\frac{1}{2}\left[\frac{\ln(1-z)}{z} + \frac{\ln(z)}{1-z}\right]. \tag{4.4.5}$$

In particular, the arguments in the claim are related as follows. Setting $\frac{z_{12}}{z_{13}} =: z$ we obtain $\frac{z_{23}}{z_{21}} = 1 - \frac{1}{z}$ and $\frac{z_{13}}{z_{23}} = \frac{1}{1-z}$. Although lengthy, with these observations the verification is straight-forward. The anti-holomorphic case will be similar. □

From the lemma the integration of the three-current correlators (4.2.19) can readily be read-off. As we have mentioned before, we want to consider pure momentum and pure winding scattering in the H- and R-flux. Since the latter is obtained by performing three T-dualities, the anti-holomorphic three-$\overline{J}$ correlator comes with a different sign. Thus we have to distinguish these two cases; the corresponding *basic three-point functions* read

$$\begin{aligned}\big\langle \mathcal{X}^a(z_1,\overline{z}_1)\,\mathcal{X}^b(z_2,\overline{z}_2)\,\mathcal{X}^c(z_3,\overline{z}_3)\big\rangle^H &= \theta^{abc}\left[\mathcal{L}\left(\tfrac{z_{12}}{z_{13}}\right) - \mathcal{L}\left(\tfrac{\overline{z}_{12}}{\overline{z}_{13}}\right)\right]\\ \big\langle \mathcal{X}^a(z_1,\overline{z}_1)\,\mathcal{X}^b(z_2,\overline{z}_2)\,\mathcal{X}^c(z_3,\overline{z}_3)\big\rangle^R &= \theta^{abc}\left[\mathcal{L}\left(\tfrac{z_{12}}{z_{13}}\right) + \mathcal{L}\left(\tfrac{\overline{z}_{12}}{\overline{z}_{13}}\right)\right]\end{aligned} \tag{4.4.6}$$

with $\theta^{abc} := \frac{(\alpha')^2}{12}\, H^{abc}$ and

$$\mathcal{L}(z) := L(z) + L(1-\frac{1}{z}) + L(\frac{1}{1-z})\,. \tag{4.4.7}$$

We abbreviated

$$\big\langle \mathcal{X}^a(z_1,\overline{z}_1)\,\mathcal{X}^b(z_2,\overline{z}_2)\,\mathcal{X}^c(z_3,\overline{z}_3)\big\rangle^R \equiv \big\langle \widetilde{\mathcal{X}}^a(z_1,\overline{z}_1)\,\widetilde{\mathcal{X}}^b(z_2,\overline{z}_2)\,\widetilde{\mathcal{X}}^c(z_3,\overline{z}_3)\big\rangle^R. \tag{4.4.8}$$

Analogously to (3.3.7), the combinatorics necessary for contracting exponentials with the perturbation can be captured by

$$\begin{aligned}&{:}\mathcal{F}(\mathcal{X}){:}\,{:}\mathcal{G}(\mathcal{X}){:}{:}\mathcal{H}(\mathcal{X}){:}\\ &\qquad = \exp\left[\int d^2z_1\, d^2z_2\, d^2z_3\; \mathfrak{G}^{abc}_{H/R}(z_1,z_2,z_3)\, \tfrac{\delta}{\delta\mathcal{X}^a_F(z_1,\overline{z}_1)}\, \tfrac{\delta}{\delta\mathcal{X}^b_G(z_2,\overline{z}_2)}\, \tfrac{\delta}{\delta\mathcal{X}^c_H(z_3,\overline{z}_3)}\right]\\ &\qquad\quad \times {:}\mathcal{F}\,\mathcal{G}\,\mathcal{H}{:}\end{aligned} \tag{4.4.9}$$

with $\mathfrak{G}^{abc}_{H/R}(z_1,z_2,z_3)$ denoting the respective correlator in (4.4.6).

Remark 11.

- Since the interaction term in (4.1.3) is of third order in the fields, evaluating correlation functions in the perturbation theory up to linear order in the flux is completely captured by substituting (4.4.2) respectively (4.4.6) for all possible two- and three-contractions.
- We omitted considering an integration constant for (4.4.6). As we will see later in section 4.5, (4.4.6) implies a Jacobi identity for the coordinates which would be non-vanishing in particular for coordinates at different points in space. Also in view of the open string analogue this seems unnatural.

4.4.2 The structure of N-point functions

The propagator (4.4.2) is the same as in the free theory. Thus, all the implications from the perturbation are encoded in the contractions evaluated using (4.4.6) and a possible phase we would like to detect can just arise from these. Given the observations of section 4.3 we consider scattering of either of the two tachyon vertex operators

$$\begin{aligned} V_i^H \equiv V_{p_i}(z_i, \overline{z}_i) &= :\exp\left[ip_i \cdot \mathcal{X}(z_i, \overline{z}_i)\right]: \\ V_i^R \equiv V_{w_i}(z_i, \overline{z}_i) &= :\exp\left[iw_i \cdot \widetilde{\mathcal{X}}(z_i, \overline{z}_i)\right]: \,. \end{aligned} \tag{4.4.10}$$

We will set $w|_H \equiv p|_R$ and write the superscripts of the vertex operators as superscript of the correlator if inserted in order to keep the notation simple. In any case, the background under consideration will be clear from the notation.

Proposition 4. *The correlation function of N tachyon vertex operators in* CFT_H *with H- respectively R-flux background reads*

$$\begin{aligned} \langle V_1\, V_2\, \ldots V_N\rangle^{H/R} = \langle V_{0,1}\, V_{0,2} \ldots V_{0,N}\rangle_0^{H/R} \times \\ \exp\left\{-i\theta^{abc} \sum_{1\le i<j<k\le N} p_{i,a}\, p_{j,b}\, p_{k,c} \left[\mathcal{L}\left(\tfrac{z_{ij}}{z_{ik}}\right) \mp \mathcal{L}\left(\tfrac{\overline{z}_{ij}}{\overline{z}_{ik}}\right)\right]\right\}_\theta \end{aligned} \tag{4.4.11}$$

with the first factor denoting the corresponding free correlator an the subscript of the second factor indicates that this has to be understood linear in θ.

Proof. By applying (4.4.9) to $:V_1^{R/H} \ldots V_N^{R/H}:$ the claim immediately follows. □

From now on we will drop the subscript θ. As before, all results have to be understood linear order in θ.

Crossing symmetry

We formulated CFT_H on the sphere S^2 whose conformal Killing group is $PSL(2,\mathbb{C})$. It acts by arbitrarily permuting the operator insertions on the sphere respectively the compactified complex plane $\mathbb{C} \cup \{\infty\}$ and has to be a symmetry in a proper conformal field theory. Thus in particular (4.4.11) has to be invariant under arbitrary permutations of the vertex operators. This property is non-trivial and in order to show that, we need the following

Lemma 7. The redefined Rogers dilogarithm (4.4.7) satisfies

$$\begin{aligned} \mathcal{L}(z) &= \mathcal{L}(1 - \tfrac{1}{z}) = \mathcal{L}(\tfrac{1}{1-z}) \\ \mathcal{L}(z) &= \tfrac{\pi^2}{2} - \mathcal{L}(1-z)\,, \end{aligned} \tag{4.4.12}$$

where $\frac{\pi^2}{6} = L(1)$ with L the usual Rogers dilogarithm (4.4.3).

Proof. The first claim is a direct consequence of the definition of $\mathcal{L}$. The second claim follows immediately from the corresponding properties of Rogers dilogarithm L, namely $L(z) + L(1-z) = L(1)$ [39]. □

In order to show $PSL(2,\mathbb{C})$ we will first consider $N = 3$. Introducing the shorthand notation

$$\{123\}^{\mp} := \exp\left\{-i\theta^{abc}\, p_{1,a}\, p_{2,b}\, p_{3,c}\left[\mathcal{L}\left(\tfrac{z_{12}}{z_{13}}\right) \mp \mathcal{L}\left(\tfrac{\bar{z}_{12}}{\bar{z}_{13}}\right)\right]\right\} \tag{4.4.13}$$

and $y := \frac{z_{12}}{z_{13}}$ we find the possibilities listed in Table 4.2. Any permutation changes

permutation	argument	$\mathcal{L}$	$\mathcal{L}+\overline{\mathcal{L}}$	sign for momenta	phase for R
$\{123\}^{\mp}$	y	$\mathcal{L}(y)$	$\times$	$+$	$\times$
$\{312\}^{\mp}$	$\frac{1}{1-y}$	$\mathcal{L}(y)$	$\times$	$+$	$\times$
$\{231\}^{\mp}$	$1-\frac{1}{y}$	$\mathcal{L}(y)$	$\times$	$+$	$\times$
$\{132\}^{\mp}$	$1-y$	$\frac{\pi^2}{2}-\mathcal{L}(y)$	π^2	$-$	$i\pi^2\theta^{abc}p_{1,a}p_{2,b}p_{3,c}$
$\{213\}^{\mp}$	$\frac{1}{y}$	$\frac{\pi^2}{2}-\mathcal{L}(y)$	π^2	$-$	$i\pi^2\theta^{abc}p_{1,a}p_{2,b}p_{3,c}$
$\{321\}^{\mp}$	$\frac{y}{y-1}$	$\frac{\pi^2}{2}-\mathcal{L}(y)$	π^2	$-$	$i\pi^2\theta^{abc}p_{1,a}p_{2,b}p_{3,c}$

Table 4.2: The way (4.4.13) changes upon permutation of the operator insertions.

$\frac{z_{\sigma(1)\sigma(2)}}{z_{\sigma(1)\sigma(3)}}$ which can be written in terms of y; this is given in the second row of the table. All resulting combinations of y as arguments of $\mathcal{L}$ can be expressed as $\mathcal{L}(y)$ by using (4.4.12), listed in the third row. A possible additional constant due to the transformations can in total be apparent only for the R-flux, i.e. "+"-case since they appear for $\mathcal{L}(y)$ and $\mathcal{L}(\bar{y}) \equiv \overline{\mathcal{L}}(y)$ likewise. This is shown in the fourth row. The fifth row shows the sign of the combinations of momenta $p_1p_2p_3$ due to the antisymmetry of θ; we observe the sign of $\mathcal{L}(y)$ for odd permutations which compensates for the sign change by θ. The last row summarizes the collected additional phase.

Thus, upon permuting the operator insertions we have found

$$\begin{aligned}\{\sigma(1)\sigma(2)\sigma(3)\}^- &= \{123\}^- \\ \{\sigma(1)\sigma(2)\sigma(3)\}^+ &= \exp\left[i\left(\tfrac{1-\mathrm{sgn}(\sigma)}{2}\right)\pi^2\,\theta^{abc}\,p_{1,a}\,p_{2,b}\,p_{3,c}\right]\{123\}^+\end{aligned} \tag{4.4.14}$$

with a phase for odd permutation *only* in the R-flux background. This seems to spoil $PSL(2,\mathbb{C})$-invariance. But as we already encountered in section 3.3, the free propagator in (4.4.11) imposes momentum conservation. In particular this implies $p_3 = -(p_1 + p_2)$. Thus, after using momentum conservation the additional phase disappears and (4.4.11) is a proper CFT correlation function, at least for $N = 3$.

Proposition 5. *Upon interchanging the the i-th and the $i+1$-th operator in* (4.4.11) *we find*

$$\langle V_1 \, \ldots V_{i+1} \, V_i \, \ldots V_N \rangle^{H/R} = \exp\left[i\epsilon\pi^2 \, \theta^{abc} \sum_{\substack{j=1 \\ j\notin\{i,i+1\}}}^{N} p_{i,a} \, p_{i+1,b} \, p_{j,c} \right] \times \langle V_1 \, \ldots V_i \, V_{i+1} \, \ldots V_N \rangle^{H/R} \tag{4.4.15}$$

with $\epsilon = 0$ for the H-flux and $\epsilon = 1$ for the R-flux for all N ans arbitrary permutations can be deduced. Furthermore the phase vanishes by using momentum conservation, i.e. (4.4.11) is $PSL(2,\mathbb{C})$-invariant.

Proof. From (4.4.11) we observe that only those triples of indices which contain i and $i+1$ are affected. The corresponding momentum terms get an extra minus sign by restoring the ordering $1 \leq i < j < k \leq N$. The arguments of $\mathcal{L}$ change as follows. With $x = \frac{z_{i(i+1)}}{z_{ij}}$ and $y = \frac{z_{ji}}{z_{j(i+1)}}$ we have

$$\frac{z_{(i+1)i}}{z_{(i+1)j}} = \frac{x}{x-1} \quad \text{for } j > i+1 \quad , \quad \frac{z_{j(i+1)}}{z_{ji}} = \frac{1}{y} \quad \text{for } j < i \, . \tag{4.4.16}$$

Using (4.4.12), i.e. $\mathcal{L}\left(\frac{x}{x-1}\right) = \frac{\pi^2}{2} - \mathcal{L}(x)$ and $\mathcal{L}\left(\frac{1}{y}\right) = \frac{\pi^2}{2} - \mathcal{L}(y)$ we obtain the desires phase indeed only for the R-flux since the $\frac{\pi^2}{2}$-terms add up. All possible permutations can be generated by a finite number of transposition. Therefore (4.4.15) suffices to deduce phases for any different operator ordering.

For proving the last assertion we proceed inductively. We have already seen $N = 3$. Suppose the phase vanishes for $N-1$ and $i+1 < N$ (without loss of generality). Then

$$\begin{aligned} \theta^{abc} \sum_{\substack{j=1 \\ j\notin\{i,i+1\}}}^{N} p_{i,a} \, p_{i+1,b} \, p_{j,c} &= \theta^{abc} \sum_{\substack{j=1 \\ j\notin\{i,i+1\}}}^{N-1} p_{i,a} \, p_{i+1,b} \, p_{j,c} + \theta^{abc} \, p_{i,a} \, p_{i+1,b} \, p_{N,c} \\ &= \theta^{abc} \sum_{\substack{j=1 \\ j\notin\{i,i+1\}}}^{N-1} p_{i,a} \, p_{i+1,b} \left(-\sum_{n\neq j}^{N} p_{n,c} \right) + \theta^{abc} \, p_{i,a} \, p_{i+1,b} \, p_{N,c} \\ &= \theta^{abc} \, p_{i,a} \, p_{i+1,b} \, p_{N,c} - \theta^{abc} \, p_{i,a} \, p_{i+1,b} \, p_{N,c} = 0 \, . \end{aligned} \tag{4.4.17}$$

Thus (4.4.11) is indeed $PSL(2,\mathbb{C})$-invariant. □

To illustrate the previous proposition it is instructive to consider again $N = 3$. For instance, we have

$$\begin{aligned} \{312\}^+ &= \exp\left(i\pi^2 \, \theta^{abc} \, p_{1,a} \, p_{3,b} \, p_{2,c}\right) \{132\}^+ \\ &= \exp\left(-i\pi^2 \, \theta^{abc} \, p_{1,a} \, p_{2,b} \, p_{3,c}\right) \{132\}^+ \\ &= \exp\left(-i\pi^2 \, \theta^{abc} \, p_{1,a} \, p_{2,b} \, p_{3,c}\right) \left[\exp\left(i\pi^2 \, \theta^{abc} \, p_{1,a} \, p_{3,b} \, p_{2,c}\right) \{123\}^+\right] \\ &= \{123\}^+ \, , \end{aligned} \tag{4.4.18}$$

confirming the respective result given in Table 4.2.

To summarize, we deduced the general form of N-point functions of tachyon vertex operators (4.4.11) and explicitly showed in proposition (5) that these are proper CFT correlation functions. We also detected a phase appearing by the comparison of different operator insertions. Although absent after imposing momentum conservation the occurrence of this phase is remarkable since it arises from non-trivial properties of Rogers dilogarithm. We will consider this the desired phase analogous to the open string case and further elaborate on it in the next section.

4.5 Emergence of nonassociative geometry

In the preceding sections we developed CFT_H and studied the structure of N-point correlation functions in the case of tachyons. This enables us to seize the strategy emphasized in chapter 3 and study the inference of CFT correlation functions on the spacetime geometry.

4.5.1 Nonassociative coordinates

The purpose of the following is the investigation of algebraic properties of the coordinates $\mathcal{X}^a$. In order for that we will employ the principle already observed in remark 4 in section 3.3 to use radial ordering to deduce commutators from propagators.

The candidate for finding new properties are the basic three-point functions (4.4.6) since the propagator (4.4.2) is not corrected linear order in the flux. Let us collect some necessary special values for $\mathcal{L}$ following from the special values $L(0) = 0$, $L(\infty) = \frac{\pi^2}{3}$ and $L(-\infty) = -\frac{\pi^2}{6}$ of Rogers dilogarithm [40] and the definition (4.4.7);

$$\mathcal{L}(0) = 0 \quad , \quad \mathcal{L}(\infty) = \tfrac{\pi^2}{2} \quad , \quad \mathcal{L}(-\infty) = 0\,. \tag{4.5.1}$$

We would like to evaluate the equal time, i.e. equal radius double-commutator of the coordinates using (4.4.6). In particular, only the equal space double-commutator is expected to reveal interesting features since for arbitrary points we cannot make any particular statements about $\mathcal{L}$. If we pick small numbers $0 < \delta_1 < \delta_2$ while introducing the shorthand notation $\mathcal{X}^a(z) \equiv \mathcal{X}^a(|z|)$ since we are interested in equal space points, i.e. the angle remains the same, we find

$$\begin{aligned}
&\big\langle R\,\big\{\big[\mathcal{X}^a(|z|),\big[\mathcal{X}^b(|z|),\mathcal{X}^c(|z|)\big]\big]\big\}\,\big\rangle^{H/R} \\
&\quad= \lim_{\delta_1,\delta_2\to 0}\big\langle \mathcal{X}^a(|z|)\,\mathcal{X}^b(|z|+\delta_1)\,\mathcal{X}^c(|z|+\delta_2) - \mathcal{X}^a(|z|)\,\mathcal{X}^b(|z|+\delta_2)\,\mathcal{X}^c(|z|+\delta_1) \\
&\qquad\quad - \mathcal{X}^a(|z|)\,\mathcal{X}^b(|z|-\delta_2)\,\mathcal{X}^c(|z|-\delta_1) + \mathcal{X}^a(|z|)\,\mathcal{X}^b(|z|-\delta_1)\,\mathcal{X}^c(|z|-\delta_2)\big\rangle^{H/R} \\
&\quad= \theta^{abc}\lim_{\delta_1,\delta_2\to 0}\big\langle\big[\mathcal{L}\left(\tfrac{+}{++}\right) - \mathcal{L}\left(\tfrac{++}{+}\right) - \mathcal{L}\left(\tfrac{--}{-}\right) + \mathcal{L}\left(\tfrac{-}{--}\right)\big] \mp \big[\text{c.c.}\big]\big\rangle \\
&\quad= \theta^{abc}\big\langle\big[\mathcal{L}(\infty) - \mathcal{L}(0) - \mathcal{L}(0) + \mathcal{L}(-\infty)\big] \mp \big[\text{c.c.}\big]\big\rangle \\
&\quad= \left(\tfrac{\pi^2}{2} \mp \tfrac{\pi^2}{2}\right)\,\theta^{abc}\big\langle \mathbb{1}\big\rangle\,.
\end{aligned} \tag{4.5.2}$$

In the second step we employed (4.4.6) where we indicated $\delta_1 \equiv +$, $\delta_2 \equiv ++$, $-\delta_1 \equiv -$ and $-\delta_2 \equiv --$. This is in order to illustrate the "speed" of convergence and whether zero is reached from above or below (e.g. $++$ is the fastest from above). The third step then applies the limit and for the last step we used (4.5.1). Thus we have found

$$\begin{aligned} \big[\,\mathcal{X}^a(|z|)\,,\big[\,\mathcal{X}^b(|z|)\,,\mathcal{X}^c(|z|)\,\big]\,\big] &= 0 \\ \big[\,\widetilde{\mathcal{X}}^a(|z|)\,,\big[\,\widetilde{\mathcal{X}}^b(|z|)\,,\widetilde{\mathcal{X}}^c(|z|)\,\big]\,\big] &= \pi^2\,\theta^{abc}\,. \end{aligned} \tag{4.5.3}$$

An algebraic property encoded by at least three objects is associativity rather than commutativity and can be characterized by the *Jacobi identity*

$$\mathrm{JI}(A,B,C) := [A,[B,C]] + [C,[A,B]] + [B,[C,A]] \tag{4.5.4}$$

for any A, B, C in some algebra. The commutators are to be understood with respect to the (ring-)multiplication. Writing out the definition it is easy to see that that a commutative as well as an associative product implies a vanishing Jacobi identity. Negating this, *a non-vanishing Jacobi identity implies the product of the algebra to be noncommutative and nonassociative.*

On the algebra of functions $C^\infty(M)$ on the spacetime $M = \mathbb{T}^3$ or $M = \mathbb{T}R^3$ we define

$$\big[\mathcal{X}^a, \mathcal{X}^b, \mathcal{X}^c\big] := \lim_{z_i \to z} \mathrm{JI}(\mathcal{X}^a(z_1), \mathcal{X}^b(z_2), \mathcal{X}^c(z_3))\,. \tag{4.5.5}$$

This can be computed using (4.5.3) for the indices cyclically permuted and we readily find

$$\begin{aligned} \big[\mathcal{X}^a, \mathcal{X}^b, \mathcal{X}^c\big] &= 0 \\ \big[\widetilde{\mathcal{X}}^a, \widetilde{\mathcal{X}}^b, \widetilde{\mathcal{X}}^c\big] &= 3\pi^2\,\theta^{abc}\,, \end{aligned} \tag{4.5.6}$$

i.e. the coordinates of the R-flux background are noncommutative and nonassociative (*NCA*).

Thus, from the basic three-point function (4.4.6) we deduced that the spacetime (non-)geometry for the R-flux background is NCA, which is opposed to the spacetime geometry of the H-flux.

This was first detected in [12] from studying the $SU(2)_k$ WZW model and parallel by [13] with a different approach.

4.5.2 The CFT perspective

The methods employed in [12] to detect the NCA (4.5.6) could have also been used. We could have expanded the coordinates $\mathcal{X}^a$ in a Laurent series with the coefficients of the chiral currents (4.2.4) appearing as modes. These obey the Kač-Moody algebra (4.2.23) with different signs for the structure constant $H^a{}_{bc}$ in the H- and R-flux case and (4.5.5) could have been worked-out more directly. However, the zero-modes elude this procedure and were treated differently in [12].

The crucial thing to observe was already emphasized in section 4.3, that is, T-duality acts on the currents (4.2.4) by

$$J^a(z) \xrightarrow{T} J^a(z) \quad , \quad \overline{J}^a(\overline{z}) \xrightarrow{T} -\overline{J}^a(\overline{z})\,. \tag{4.5.7}$$

Considering three T-dualities as we did above therefore changes

- the sign of the structure constant in the anti-holomorphic copy of the Kač-Moody algebra (4.2.23)

  H-**flux:** $[\,\overline{j}^a_m, \overline{j}^b_n\,] = +i\frac{\alpha'}{4}\, H^{ab}{}_c\, \overline{j}^c_{m+n} + m\,\delta^{ab}\,\delta_{m,-n}$

  R-**flux:** $[\,\overline{j}^a_m, \overline{j}^b_n\,] = -i\frac{\alpha'}{4}\, H^{ab}{}_c\, \overline{j}^c_{m+n} + m\,\delta^{ab}\,\delta_{m,-n}$

- the sign of the anti-holomorphic three point function in (4.2.19)

and also establishes the difference between the basic three-point function (4.4.6) for H- and R-flux. Thus all features of the R-flux background absent for the H-flux, in particular NCA (4.5.6) attribute to the different sings for the currents (4.2.4). To obtain a highest-weight representation of the algebra (4.2.23) one defines *raising* and *lowering* operators

$$J^3_n := -\kappa\, j^3_n \quad , \quad J^\pm_n := \kappa\left(j^1_n \pm i\, j^2_n\right) \tag{4.5.8}$$

with $\kappa = -\frac{4}{\alpha' H^{123}}$ as new basis since they satisfy the commutation relations desired constructing these representations [35]:

$$\left[J^3_m, J^3_n\right] = \kappa^2\, m\,\delta_{m,-n}\,,\ \left[J^3_m, J^\pm_n\right] = \pm J^\pm_{m+n}\,,\ \left[J^+_m, J^-_n\right] = 2\kappa^2\, m\,\delta_{m,-n} + 2\, J^3_{m+n}\,. \tag{4.5.9}$$

The same can be done for the second copy. Thus the Kač-Moody algebra for the fluxes can be generated by

$$\begin{aligned} H - \text{flux} \ &: \ \left\{J^3_n\,,\ J^\pm_n = \kappa\left(j^1_n \pm i\, j^2_n\right)\right\} \times \left\{-\overline{J}^3_n\,,\ -\overline{J}^\pm_n = -\kappa\left(\overline{j}^1_n \pm i\,\overline{j}^2_n\right)\right\} \\ R - \text{flux} \ &: \ \left\{J^3_n\,,\ J^\pm_n = \kappa\left(j^1_n \pm i\, j^2_n\right)\right\} \times \left\{\overline{J}^3_n\,,\ \overline{J}^\pm_n = \kappa\left(\overline{j}^1_n \pm i\,\overline{j}^2_n\right)\right\}\,. \end{aligned} \tag{4.5.10}$$

With these definitions one can check that the H- and R-flux case yield the same algebra (4.5.9) upon using the respective Kač-Moody algebras (4.2.23). One therefore obtains the same representations etc. for the CFT corresponding to the H- and R-flux background and we can infer that

> From the point of view of conformal field theory, the H- and R-flux backgrounds are indistinguishable although the spacetime geometries are very different.

Remark 12. It is interesting to note that the different relative sign between the holomorphic currents (4.2.4) for the R-flux is captured by the symmetry of the WZW model on a Lie group G discussed in subsection 2.2.2 since $G(\overline{z}) = G^{-1}(\overline{z})$ as a group (with multiplication as group structure). We already mentioned that the usual T-dual theories are equivalent as CFT's on the quantum level. The last observations hint towards the possibility of extending this statement also to the R-flux case which cannot be obtained by applying the Buscher rules.

4.5.3 An N-ary product

So far we found a worldsheet independent phase (4.4.15) and also nonassociative coordinates (4.5.6) only on the spacetime manifold for the R-flux background. Following section 3.3 we can encode the phase by defining the *triangle-N-product*

$$\boxed{\begin{aligned}&(f_1 \vartriangle_N f_2 \vartriangle_N \ldots \vartriangle_N f_N)(x) := \\ &\qquad \exp\Big[\frac{\pi^2}{2}\,\theta^{abc} \sum_{\substack{i,j,k=1\\ i<j<k}}^{N} \frac{\partial}{\partial x_i^a}\,\frac{\partial}{\partial x_j^b}\,\frac{\partial}{\partial x_k^c}\Big]\, f_1(x_1)\, f_2(x_2) \ldots f_N(x_N)\Big|_{x_i = x}\end{aligned}} \tag{4.5.11}$$

for $f_i \in C^\infty(\mathbb{T}R^3)$. This is the closed string generalization of the open string noncommutative product (3.3.11).

To show that this indeed captures the possible phases it suffices to show it for the transposition in (4.4.15). We have

$$\begin{aligned}
&(f_1 \vartriangle_N \ldots \vartriangle_N f_{n+1} \vartriangle_N f_n \vartriangle_N \ldots \vartriangle_N f_N)(x) \\
&\quad \simeq \exp\Big[\tfrac{\pi^2}{2}\,\theta^{abc} \sum_{\substack{i,j,k=1\\ i<j<k}}^{N} \tfrac{\partial}{\partial x_i^a}\,\tfrac{\partial}{\partial x_j^b}\,\tfrac{\partial}{\partial x_k^c}\Big] \\
&\quad = \exp\Big[\tfrac{\pi^2}{2}\,\theta^{abc} \sum_{i,j,k\notin\{n,n+1\}}^{N} \tfrac{\partial}{\partial x_i^a}\,\tfrac{\partial}{\partial x_j^b}\,\tfrac{\partial}{\partial x_k^c}\Big] \exp\Big[\tfrac{\pi^2}{2}\,\theta^{abc} \sum_{i\neq n,n+1}^{N} \tfrac{\partial}{\partial x_{n+1}^a}\,\tfrac{\partial}{\partial x_n^b}\,\tfrac{\partial}{\partial x_i^c}\Big] \\
&\quad = \exp\Big[\tfrac{\pi^2}{2}\,\theta^{abc} \sum_{i,j,k\notin\{n,n+1\}}^{N} \tfrac{\partial}{\partial x_i^a}\,\tfrac{\partial}{\partial x_j^b}\,\tfrac{\partial}{\partial x_k^c}\Big] \exp\Big[-\tfrac{\pi^2}{2}\,\theta^{abc} \sum_{i\neq n,n+1}^{N} \tfrac{\partial}{\partial x_n^a}\,\tfrac{\partial}{\partial x_{n+1}^b}\,\tfrac{\partial}{\partial x_i^c}\Big] \\
&\quad = \exp\Big[-\pi^2\,\theta^{abc} \sum_{i\neq n,n+1}^{N} \tfrac{\partial}{\partial x_n^a}\,\tfrac{\partial}{\partial x_{n+1}^b}\,\tfrac{\partial}{\partial x_i^c}\Big] \exp\Big[\tfrac{\pi^2}{2}\,\theta^{abc} \sum_{\substack{i,j,k=1\\ i<j<k}}^{N} \tfrac{\partial}{\partial x_i^a}\,\tfrac{\partial}{\partial x_j^b}\,\tfrac{\partial}{\partial x_k^c}\Big] \\
&\quad \simeq \exp\Big[-\pi^2\,\theta^{abc} \sum_{i\neq n,n+1}^{N} \tfrac{\partial}{\partial x_n^a}\,\tfrac{\partial}{\partial x_{n+1}^b}\,\tfrac{\partial}{\partial x_i^c}\Big] \\
&\qquad\qquad\qquad \times (f_1 \vartriangle_N \ldots \vartriangle_N f_n \vartriangle_N f_{n+1} \vartriangle_N \ldots \vartriangle_N f_N)(x)
\end{aligned} \tag{4.5.12}$$

where $\simeq$ means that we just indicated the exp-structure. Since the derivatives commute, the operator-phase can be put in front of the differential operator in the definition (4.5.11). Choosing $f_i(x) = \exp(ip_i \cdot x)$ the action of the operator-phase gives the desired phase in (4.4.15) and thus encodes the properties of (at least) tachyon correlators upon permutations.

The triangle-N-product has the following properties.

- Using $\triangle_3 \equiv \triangle$ we can calculate
$$\begin{aligned}[x^a, x^b, x^c] :&= \sum_{\sigma\in\Sigma_3} \mathrm{sign}(\sigma)\, x^{\sigma(a)} \triangle\, x^{\sigma(b)} \triangle\, x^{\sigma(c)} \\ &= \sum_{\sigma\in\Sigma_3} \mathrm{sign}(\sigma)\, \tfrac{\pi^2}{2}\, \theta^{\sigma(a)\sigma(b)\sigma(c)} = 3\pi^2\end{aligned} \tag{4.5.13}$$
where we used that Σ_3 contains three even and three odd permutations and that θ is totally antisymmetric[5]. Therefore the triangle-tri-product implies the non-vanishing Jacobi identity (4.5.6).

- As opposed to (3.3.11), the triangle-N-product (4.5.11) cannot be obtained by successive application of the tri-product. This can already be seen in the easiest instance;
$$\begin{aligned}[(f_1 \triangle f_2 \triangle f_3) \triangle f_4 \triangle f_5\,](x) &= (f_1 \ldots f_5)(x) + \tfrac{\pi^2}{2}\theta^{abc}\,[\,\partial_a f_1\, \partial_b f_2\, \partial_c f_3\, f_4\, f_5 \\ &\qquad + \partial_a(f_1\, f_2\, f_3)\, \partial_b f_4\, \partial_c f_5\,](x) + \mathcal{O}(\theta^2) \\ &\neq (f_1 \triangle_5 f_2 \triangle_5 f_3 \triangle_5 f_4 \triangle_5 f_5)(x)\end{aligned} \tag{4.5.14}$$
as the case where one derivative acts on either f_4 or f_5 misses. This can be seen as intrinsically reflecting the nonassociativity of (4.5.11).

- The triangle-products of different order can however be related via
$$f_1 \triangle_N \ldots \triangle_N f_{N-1} \triangle_N 1 = f_1 \triangle_{N-1} \ldots \triangle_{N-1} f_{N-1} \tag{4.5.15}$$
which can be readily verified. In principle, this also allows for defining a two-product; $f \triangle_2 g = f \triangle_3 g \triangle_3 1 = f\, g$ – the usual commutative product. However, we know that this can't be the correct one since we argued that $\mathbb{T}R^3$ is in particular noncommutative.

To summarize, we constructed an N-ary product (4.5.11) on the spacetime $\mathbb{T}R^3$ for the constant R-flux background from which one can deduce the NCA (4.5.6) and also the phase (4.4.15) which appears when permuting the operator insertions. This completes our aim of generalizing the open string analysis in chapter 3 – which revealed noncommutative geometry on D-branes – to the case of closed strings with constant H-respectively R-flux background. For the latter we detected NCA which completes the picture in that the boundary of the string (if any) detects noncommutative geometry and the bulk detects NCA geometry for the simples B-fields turned-on.

[5]Note that $[x^a, x^b, x^c]$ is just the Jacobi identity.

Chapter 5

Conclusion

After having introduced basic notions for strings moving in background fields, we reviewed noncommutative geometry in open string theory to emphasize the guiding principle for the main part of the thesis. There we studied the structure of closed bosonic string theory probing the geometric background of a constant H-flux along a flat three-torus and probing the non-geometric R-flux background obtained after applying three formal T-dualities to the former.

The main concern was to generalize the open string noncommutativity for a constant B-field background to the bulk and to characterize the spacetime topology by a product deduced from CFT correlation functions. We developed CFT_H as the proper conformal field theory describing the constant H-flux configuration up to linear order in H. For CFT_H, we identified chiral currents analogous to the currents in WZW models and computed two- and three-point functions of them. From these, the OPE's of the currents were deduced, which revealed, as expected, the same current algebra as for WZW models with H appearing as structure constant. The three-point functions were integrated to give the basic three-point function for the coordinates $\mathcal{X}^a$; they crucially depend on Rogers dilogarithm. From the point of view of conformal field theory, T-duality acts particularly simple as the right-moving parts of the coordinates just get reflected. This enabled us also to consider the basic three-point function for the T-dual coordinate $\widetilde{\mathcal{X}}^a$ describing the R-flux background. The two three-point functions just differ by a sign between the holomorphic and anti-holomorphic constituents. However, this sign turned out to imply a non-vanishing Jacobi identity only for the coordinates of the R-flux. Thus we found the R-flux background to be noncommutative and nonassociative, confirming the result in [12]. This explicitly identifies the R-flux as origin for nonassociative geometry in closed string theory.

Furthermore, we studied tachyon vertex operators in CFT_H which turned out to be physical objects but which only carry the usual center-of-mass momentum and winding quantum numbers upon imposing $\vec{p}\times\vec{w}=\vec{0}$. Thus we could only reliably compute pure momentum scattering in the H- or R-flux background, where the latter is defined by pure winding scattering in the former. Very generally, we computed N-point functions of tachyons in the H- and R-flux background by using the basic three point functions and the uncorrected propagator.

We detected a non-trivial phase only for the R-flux by permuting operator insertions which relied on the properties of Rogers dilogarithm. To reconcile with $PSL(2,\mathbb{C})$-invariance, we observed that the phase vanishes after imposing momentum conservation. However, the appearance of this phase reflects a genuine spacetime property since it is independent of the worldsheet coordinates and we suppose that the somewhat "light" appearance just reflects the simplicity of our background. A more complicated analysis incorporating back reactions on the geometry, i.e. going beyond linear order in the fluxes may give a more apparent phase.

Analogous to the open string case, we defined a nonassociative product for functions on the space corresponding to the R-flux capturing the phase. Although a straightforward generalization of the Moyal star-product, its properties are very different. Most importantly, the new N-product can *not* be obtained by successive application of the lower order products.

Since it already appeared in the simplest cases, the example of the open string noncommutativity and the closed string NCA show that generic string backgrounds have these more complicated (non-)geometries. The usual commutative and associative geometries are rather an exception.

Of course, lots of interesting issues worth studying arise.

- Generalizing the analysis to the superstring, in particular the construction of SCFT_H.
- The OPE's we computed show logarithmic terms which may hints towards a logarithmic CFT.
- We have seen that the consistency equations for string backgrounds arise as equations of motion of Einstein-Cartan theory. Since NCA geometries seem to be more generic than the usual ones, we may ask if it is possible to reconstruct the full consistency equations to all orders in α' from a pure target space approach via an nonassociative Einstein gravity.
- From a more mathematical point of view, it would be interesting to study geometry with the nonassociative product we proposed on its own right.

Bibliography

[1] H. S. Snyder, "Quantized space-time," *Phys.Rev.* **71** (1947) 38–41.

[2] A. Connes, *Noncommutative geometry*. Academic Press, 1994.

[3] J. Madore, *An introduction to noncommutative differential geometry and its physical applications*. London Mathematical Society lecture note series. Cambridge University Press, 1999.

[4] R. J. Szabo, "Quantum field theory on noncommutative spaces," *Phys.Rept.* **378** (2003) 207–299, `arXiv:hep-th/0109162 [hep-th]`.

[5] E. Braaten, T. L. Curtright, and C. K. Zachos, "Torsion and Geometrostasis in Nonlinear Sigma Models," *Nucl.Phys.* **B260** (1985) 630.

[6] A. Giveon, M. Porrati, and E. Rabinovici, "Target space duality in string theory," *Phys.Rept.* **244** (1994) 77–202, `arXiv:hep-th/9401139 [hep-th]`.

[7] M. Rocek and E. P. Verlinde, "Duality, quotients, and currents," *Nucl.Phys.* **B373** (1992) 630–646, `arXiv:hep-th/9110053 [hep-th]`.

[8] C.-S. Chu and P.-M. Ho, "Noncommutative open string and D-brane," *Nucl.Phys.* **B550** (1999) 151–168, `arXiv:hep-th/9812219 [hep-th]`.

[9] V. Schomerus, "D-branes and deformation quantization," *JHEP* **9906** (1999) 030, `arXiv:hep-th/9903205 [hep-th]`.

[10] N. Seiberg and E. Witten, "String theory and noncommutative geometry," *JHEP* **9909** (1999) 032, `arXiv:hep-th/9908142 [hep-th]`.

[11] E. Witten, "Nonabelian Bosonization in Two-Dimensions," *Commun.Math.Phys.* **92** (1984) 455–472.

[12] R. Blumenhagen and E. Plauschinn, "Nonassociative Gravity in String Theory?," *J.Phys.A* **A44** (2011) 015401, `arXiv:1010.1263 [hep-th]`.

[13] D. Lust, "T-duality and closed string non-commutative (doubled) geometry," *JHEP* **1012** (2010) 084, `arXiv:1010.1361 [hep-th]`.

[14] J. Polchinski, "Dirichlet Branes and Ramond-Ramond charges," *Phys.Rev.Lett.* **75** (1995) 4724–4727, arXiv:hep-th/9510017 [hep-th].

[15] S. Kachru, M. B. Schulz, P. K. Tripathy, and S. P. Trivedi, "New supersymmetric string compactifications," *JHEP* **0303** (2003) 061, arXiv:hep-th/0211182 [hep-th].

[16] J. Shelton, W. Taylor, and B. Wecht, "Nongeometric Flux Compactifications," *JHEP* **10** (2005) 085, arXiv:hep-th/0508133.

[17] B. Wecht, "Lectures on Nongeometric Flux Compactifications," *Class. Quant. Grav.* **24** (2007) S773–S794, arXiv:0708.3984 [hep-th].

[18] C. M. Hull, "A geometry for non-geometric string backgrounds," *JHEP* **10** (2005) 065, arXiv:hep-th/0406102.

[19] P. Bouwknegt, K. Hannabuss, and V. Mathai, "Nonassociative tori and applications to T-duality," *Commun.Math.Phys.* **264** (2006) 41–69, arXiv:hep-th/0412092 [hep-th].

[20] R. Blumenhagen, A. Deser, D. Lust, E. Plauschinn, and F. Rennecke, "Non-geometric Fluxes, Asymmetric Strings and Nonassociative Geometry," arXiv:1106.0316 [hep-th]. * Temporary entry *.

[21] J. Polchinski, *String Theory. Vol. 1: An Introduction to the Bosonic String*. Cambridge University Press, 1998.

[22] L. Alvarez-Gaume, D. Z. Freedman, and S. Mukhi, "The Background Field Method and the Ultraviolet Structure of the Supersymmetric Nonlinear Sigma Model," *Ann. Phys.* **134** (1981) 85.

[23] C. G. Callan, Jr. and L. Thorlacius, "Sigma Models and String Theory,". Print-89-0232 (Princeton).

[24] M. A. Kervaire and J. W. Milnor, "Groups of homotopy spheres: I," *The Annals of Mathematics* **77** no. 3, (1963) pp. 504–537. http://www.jstor.org/stable/1970128.

[25] C. G. Callan, Jr., E. J. Martinec, M. J. Perry, and D. Friedan, "Strings in Background Fields," *Nucl. Phys.* **B262** (1985) 593.

[26] K. Gawedzki, "Conformal field theory: A Case study," arXiv:hep-th/9904145 [hep-th].

[27] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal field theory*. Springer, 1997.

[28] T. Buscher, "A Symmetry of the String Background Field Equations," *Phys.Lett.* **B194** (1987) 59.

[29] E. Alvarez, L. Alvarez-Gaume, J. Barbon, and Y. Lozano, "Some global aspects of duality in string theory," *Nucl.Phys.* **B415** (1994) 71–100, `arXiv:hep-th/9309039 [hep-th]`.

[30] I. T. Ellwood, "NS-NS fluxes in Hitchin's generalized geometry," *JHEP* **0712** (2007) 084, `arXiv:hep-th/0612100 [hep-th]`.

[31] L. Cornalba and R. Schiappa, "Nonassociative star product deformations for D-brane world volumes in curved backgrounds," *Commun.Math.Phys.* **225** (2002) 33–66, `arXiv:hep-th/0101219 [hep-th]`.

[32] P. Dirac, *Lectures on quantum mechanics*. Dover books on physics. Dover Publications, 2001.

[33] C.-S. Chu and P.-M. Ho, "Constrained quantization of open string in background B field and noncommutative D-brane," *Nucl.Phys.* **B568** (2000) 447–456, `arXiv:hep-th/9906192 [hep-th]`.

[34] M. Kontsevich, "Deformation quantization of Poisson manifolds. 1.," *Lett.Math.Phys.* **66** (2003) 157–216, `arXiv:q-alg/9709040 [q-alg]`.

[35] R. Blumenhagen and E. Plauschinn, *Introduction to conformal field theory*, vol. 779 of *Lect.Notes Phys.* Springer, 2009.

[36] D. Tong, "String Theory," `arXiv:0908.0333 [hep-th]`.

[37] M. Flohr, "Bits and pieces in logarithmic conformal field theory," *Int. J. Mod. Phys.* **A18** (2003) 4497–4592, `arXiv:hep-th/0111228`.

[38] L. Lewin, *Polylogarithms and associated functions*. North Holland, 1981.

[39] D. Zagier, "The Dilogarithm function in Geometry and Number Theory," *Tata Inst. Fund.Res.Stud.Math.* **12** (1988) 231–249.

[40] A. N. Kirillov, "Dilogarithm identities," *Prog. Theor. Phys. Suppl.* **118** (1995) 61–142, `arXiv:hep-th/9408113`.

Acknowledgments

I am grateful to my supervisor Priv.–Doz. Dr. Ralph Blumenhagen in many respects. I would like to thank him for letting me participate in this very interesting and intense project and for sharing some of his knowledge. I appreciate the nice and productive working atmosphere and also his openness for questions and discussion. I would also like to thank Prof. Dr. Dieter Lüst who found the time for discussion beside his many duties. My special thank is dedicated to my dear fellow Andreas Deser with whom I spent most of the time in this project having lots of fertile discussions. I would also like to thank the members of the string theory group at the MPI, first and foremost Dr. Benjamin Jurke, Thorsten Rahn and Dr. Oliver Schlotterer for support in many aspects.

Last but not least I am grateful to Dr. Robert Helling being the soul of the TMP program which gave me the opportunity to study mathematical physics at the LMU.

Hiermit erkläre ich, Felix Rennecke, die vorliegende Master-Arbeit selbstständig angefertigt und alle verwendeten Quellen nach bestem Wissen und Gewissen deutlich gekennzeichnet zu haben.

Ort, Datum Unterschrift